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Relativistic Radiation Hydrodynamic Equations in Cylindrical Coordinates

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In many astrophysical systems such as accretion disks and jets, radiation interacts with relativistically moving matter. With applications to such systems in mind, I use the covariant tensor conservation laws to derive special relativistic, time-dependent, three-dimensional energy and momentum equations for matter and radiation in cylindrical coordinates. The equations can be conveniently applied to various radiation hydrodynamic processes with cylindrical symmetry. Radiation moments, like the radiation energy density, flux, and pressure, are defined in the comoving (with the flow) frame first and then transformed to the corresponding covariant quantities. The interaction between matter and radiation is also described in the comoving frame while the equations are represented in coordinates that are fixed with respect to the central object. As a concrete example, the relativistic equations of motion for a cylindrical gas flow interacting with a spherically symmetric radiation field are presented.

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I. INTRODUCTION

Matter and radiation are generally the most important components of many astrophysical systems. Photons are produced, absorbed, and scattered by matter, and through such processes, radiation imparts energy and momentum to matter. The dynamics of matter and the characteristics of radiation should be treated simultaneously and consistently.

When matter is moving in addition, photons experience red- and blueshifts, aberration, bulk Comptonization, and so forth. Moreover, clocks tick and lengths contract differently for different observers moving with different velocities. Thus, one needs to be careful describing matter and radiation for different frames of reference. Thomas derived a special relativistic theory of radiative transfer that incorporated such effects [1]. Lindquist generalized the theory to curved spacetimes and presented the radiation moment equations for spherically symmetric cases [2]. Anderson & Spiegel [3] also derived the generalized moment equations, and Thorne [4] formulated the projected symmetric trace-free moment formalism that led to general relativistic moment equations up to an arbitrary order. Most of these work were based on comoving descriptions because the comoving frame is the most natural frame to describe the radiation field and its interactions with matter. However, the comoving-frame formalism leads to rather involved equations because the velocity of matter changes from place to place and from time to time.

I reformulated the relativistic radiation hydrodynamics from a covariant tensor description of matter and radiation for one-dimensional general-relativistic spherical flows [5, 6] and for three-dimensional special-relativistic flows in spherical coordinates [7]. In my formalism, radiation and its interactions with matter are first described by comoving quantities and then recast into covariant quantities while the radiation moment equations are presented in fixed coordinates, which make the equations easier to understand and apply. In this paper, I will show you how I can apply such a mixed-frame formalism to obtain the three-dimensional special relativistic radiation hydrodynamic equations in cylindrical coordinates, which can be useful in dealing with axisymmetric accretion disks and jets [8–12]. No cylindrical symmetry, however, is assumed for matter or radiation; therefore, the result can be applied to any three-dimensional systems. I also specifically apply the equations to the case of a spherically symmetric radiation field produced by a point light source at the center to get the equations of motion for the gas in such systems. I closely follow Ref. [7] for all definitions and derivations.

II. TENSOR EQUATIONS

1. Matter

The energy-momentum tensor of an ideal gas is

$$T^{\alpha\beta} \equiv \omega_g U^\alpha U^\beta + P_g g^{\alpha\beta}, \tag{1}$$

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where U^α is the four velocity of the gas and $\omega_g \equiv \varepsilon_g + P_g$ is the gas enthalpy per unit proper volume, which is the sum of the gas energy density ε_g and the gas pressure P_g. The enthalpy of a gas is a function of the gas temperature and density. For a non-ideal gas or other equation of state, the corresponding energy-momentum tensor should be used.

2. Radiation

The radiation stress tensor consists of zeroth, first, and second moments in the angle of the radiation field,

$$R^{\alpha\beta} = \iint I(\mathbf{n}, \nu) n^\alpha n^\beta d\nu d\Omega, \tag{2}$$

where $n^\alpha \equiv p^\alpha / h\nu$, with p^α being the four-momentum of photons, and $I(x^\alpha; \mathbf{n}, \nu)$ is the specific intensity of photons moving in direction $\mathbf{n}$ on a unit sphere of projected tangent space with the frequency ν measured by a fiducial observer.

3. Radiation Hydrodynamic Equations

The particle number, rather than the mass density, is conserved in relativistic hydrodynamics,

$$(nU^\alpha)_{;\alpha} = 0. \tag{3}$$

In the presence of an external force, the energy-momentum tensor of matter plus the stress tensor of radiation satisfy

$$\left(T^{\alpha\beta} + R^{\alpha\beta}\right)_{;\beta} = f^\alpha. \tag{4}$$

The four-force density of an external non-radiative force, $f^\alpha = (f^t, \mathbf{f})$, such as gravity, has three spatial components $\mathbf{f}$, a force density per unit proper volume, and a time component $f^t = \mathbf{v} \cdot \mathbf{f}$, where $\mathbf{v}$ is the proper velocity [13].

If the micro-physical processes of interactions between radiation and matter are known, the equation can be separated into two conservation equations, one for the radiation and one for the matter. Such interactions can be incorporated in the radiation four-force density, defined to describe the energy and the momentum transferred from the matter to the radiation [13],

$$G^\alpha \equiv \frac{1}{c} \int d\nu \int d\Omega [\chi I(\mathbf{n}, \nu) - \eta] n^\alpha, \tag{5}$$

where χ is the opacity per unit proper length and η the emissivity per proper unit volume. This radiation four-force density is equal to the divergence of the energy-momentum tensor for matter minus the external four-force density,

$$T^{\alpha\beta}{}_{;\beta} = G^\alpha + f^\alpha, \tag{6}$$

and to minus the divergence of the radiation stress tensor,

$$R^{\alpha\beta}{}_{;\beta} = -G^\alpha. \tag{7}$$

III. CYLINDRICAL COORDINATES

The above tensor equations can be recast into explicit hydrodynamic and radiation moment equations once a specific coordinate system is chosen.

1. Metric and Tetrads

A flat spacetime metric in cylindrical coordinates is

$$d\tau^2 = -g_{\alpha\beta} dx^\alpha dx^\beta = dt^2 - dR^2 - R^2 d\theta^2 - dz^2. \tag{8}$$

The Greek indices, such as α or β, run from 0 to 3 while the roman indices, such as i and j, run from 1 to 3.

Two relevant velocities can be defined. The four-velocity of the gas,

$$U^\alpha \equiv \frac{dx^\alpha}{d\tau}, \tag{9}$$

is a covariant vector, which is defined for given coordinates and satisfies the normalization condition

$$U_\alpha U^\alpha = -1. \tag{10}$$

The proper velocity, a velocity measured by a fiducial observer at rest with respect to the fixed coordinates, is a spatial three vector,

$$v^i = v_i = \frac{\tilde{u}^i}{\gamma}, \tag{11}$$

where

$$\tilde{u}^i \equiv \sqrt{g_{ii}}\, U^i = \sqrt{g^{ii}}\, U_i \equiv \tilde{u}_i \quad \text{(no summation for } i\text{)}. \tag{12}$$

The Lorentz factor for the proper velocity is $\gamma \equiv [1 - v^2]^{-1/2} = U^t$, where $v^2 \equiv v^i v_i$.

A tetrad is a locally inertial frame, and physical quantities, such as the energy density or the radiation moments, are straightforwardly defined in a given tetrad. In radiation hydrodynamics, fixed and comoving tetrads are the most relevant ones. A fixed tetrad with base $\mathbf{e}_{\hat{i}}$ is a tetrad fixed with respect to the coordinates and can be expressed in terms of the coordinate base as

$$\frac{\partial}{\partial \hat{t}} = \frac{\partial}{\partial t}, \quad \frac{\partial}{\partial \hat{R}} = \frac{\partial}{\partial R}, \quad \frac{\partial}{\partial \hat{\theta}} = \frac{1}{R}\frac{\partial}{\partial \theta}, \quad \frac{\partial}{\partial \hat{z}} = \frac{\partial}{\partial z}. \tag{13}$$

The comoving tetrad moves with velocity v^i relative to the fixed tetrad and, therefore, is related to the fixed tetrad by the Lorentz tranformation

$$\frac{\partial}{\partial x^{\hat{\alpha}}_{co}} = \Lambda^{\hat{\beta}}{}_{\hat{\alpha}}(\mathbf{v}) \frac{\partial}{\partial x^{\hat{\beta}}}, \tag{14}$$

where $\hat{\alpha}$ and $\hat{\beta}$ denote the tetrad's bases. Hence, the comoving tetrad in terms of the coordinate base is

$$\begin{aligned}
\frac{\partial}{\partial \hat{t}_{co}} &= \gamma\frac{\partial}{\partial t} + \gamma v^R \frac{\partial}{\partial R} + \gamma v^\theta \frac{1}{R}\frac{\partial}{\partial \theta} + \gamma v^z \frac{\partial}{\partial z}, \\
\frac{\partial}{\partial \hat{R}_{co}} &= \gamma v^R \frac{\partial}{\partial t} + \left[1 + (\gamma-1)\frac{v^R v_R}{v^2}\right]\frac{\partial}{\partial R} \\
&\quad + (\gamma-1)\frac{v^R v_\theta}{v^2}\frac{1}{R}\frac{\partial}{\partial \theta} + (\gamma-1)\frac{v^R v_z}{v^2}\frac{\partial}{\partial z}, \\
\frac{\partial}{\partial \hat{\theta}_{co}} &= \gamma v^\theta \frac{\partial}{\partial t} + (\gamma-1)\frac{v^\theta v_R}{v^2}\frac{\partial}{\partial R} \\
&\quad + \left[1 + (\gamma-1)\frac{v^\theta v_\theta}{v^2}\right]\frac{1}{R}\frac{\partial}{\partial \theta} + (\gamma-1)\frac{v^\theta v_z}{v^2}\frac{\partial}{\partial z}, \\
\frac{\partial}{\partial \hat{z}_{co}} &= \gamma v^z \frac{\partial}{\partial t} + (\gamma-1)\frac{v^z v_R}{v^2}\frac{\partial}{\partial R} \\
&\quad + (\gamma-1)\frac{v^z v_\theta}{v^2}\frac{1}{R}\frac{\partial}{\partial \theta} + \left[1 + (\gamma-1)\frac{v^z v_z}{v^2}\right]\frac{\partial}{\partial z}.
\end{aligned} \tag{15}$$

The inverse transformation from the comoving tetrad to the fixed tetrad is similarly obtained from the inverse Lorentz transformation $\Lambda^{\hat{\alpha}}{}_{\hat{\beta}}(-\mathbf{v})$, which also provides the transformation between the coordinate base and the comoving tetrad:

$$\begin{aligned}
\frac{\partial}{\partial \hat{t}} &= \frac{\partial}{\partial t} = \gamma\frac{\partial}{\partial \hat{t}_{co}} - \gamma v^R \frac{\partial}{\partial \hat{R}_{co}} - \gamma v^\theta \frac{\partial}{\partial \hat{\theta}_{co}} - \gamma v^z \frac{\partial}{\partial \hat{z}_{co}}, \\
\frac{\partial}{\partial \hat{R}} &= \frac{\partial}{\partial R} = -\gamma v^R \frac{\partial}{\partial \hat{t}_{co}} + \left[1 + (\gamma-1)\frac{v^R v_R}{v^2}\right]\frac{\partial}{\partial \hat{R}_{co}} \\
&\quad + (\gamma-1)\frac{v^R v_\theta}{v^2}\frac{\partial}{\partial \hat{\theta}_{co}} + (\gamma-1)\frac{v^R v_z}{v^2}\frac{\partial}{\partial \hat{z}_{co}}, \\
\frac{\partial}{\partial \hat{\theta}} &= \frac{1}{R}\frac{\partial}{\partial \theta} = -\gamma v^\theta \frac{\partial}{\partial \hat{t}_{co}} + (\gamma-1)\frac{v^\theta v_R}{v^2}\frac{\partial}{\partial \hat{R}_{co}} \\
&\quad + \left[1 + (\gamma-1)\frac{v^\theta v_\theta}{v^2}\right]\frac{\partial}{\partial \hat{\theta}_{co}} + (\gamma-1)\frac{v^\theta v_z}{v^2}\frac{\partial}{\partial \hat{z}_{co}}, \\
\frac{\partial}{\partial \hat{z}} &= \frac{\partial}{\partial z} = -\gamma v^z \frac{\partial}{\partial \hat{t}_{co}} + (\gamma-1)\frac{v^z v_R}{v^2}\frac{\partial}{\partial \hat{R}_{co}} \\
&\quad + (\gamma-1)\frac{v^z v_\theta}{v^2}\frac{\partial}{\partial \hat{\theta}_{co}} + \left[1 + (\gamma-1)\frac{v^z v_z}{v^2}\right]\frac{\partial}{\partial \hat{z}_{co}}.
\end{aligned} \tag{16}$$

2. Radiation Moments

The zeroth moments of the radiation field, the radiation energy densities, in the fixed and in the comoving tetrad frames are

$$E = \iint I_\nu d\nu d\Omega, \quad E_{co} = \iint I_{\nu_{co}} d\nu_{co} d\Omega_{co}, \tag{17}$$

where $d\Omega$ and $d\Omega_{co}$ are the solid angle elements in the fixed and the comoving tetrads, respectively [7]. The first moment, the radiation flux, has three spatial components defined as

$$F^i = \iint I_\nu n^i d\nu d\Omega, \quad F^i_{co} = \iint I_{\nu_{co}} n^i_{co} d\nu_{co} d\Omega_{co}. \tag{18}$$

The second moment, the radiation pressure tensor, is symmetric and has six components,

$$\begin{aligned}
P^{ij} &= \iint I_\nu n^i n^j d\nu d\Omega, \\
P^{ij}_{co} &= \iint I_{\nu_{co}} n^i_{co} n^j_{co} d\nu_{co} d\Omega_{co}.
\end{aligned} \tag{19}$$

The fixed tetrad components of the radiation stress tensor in terms of these moments are

$$R^{\hat{\alpha}\hat{\beta}} = \begin{pmatrix} E & F^R & F^\theta & F^z \\ F^R & P^{RR} & P^{R\theta} & P^{Rz} \\ F^\theta & P^{R\theta} & P^{\theta\theta} & P^{\theta z} \\ F^z & P^{R\phi} & P^{\theta z} & P^{zz} \end{pmatrix}, \tag{20}$$

and the comoving tetrad components are

$$R^{\hat{\alpha}\hat{\beta}}_{co} = \begin{pmatrix} E_{co} & F^R_{co} & F^\theta_{co} & F^z_{co} \\ F^R_{co} & P^{RR}_{co} & P^{R\theta}_{co} & P^{R\phi}_{co} \\ F^\theta_{co} & P^{R\theta}_{co} & P^{R\theta}_{co} & P^{\theta z}_{co} \\ F^z_{co} & P^{R\phi}_{co} & P^{\theta z}_{co} & P^{zz}_{co} \end{pmatrix}. \tag{21}$$

Meanwhile, the contravariant form of the radiation stress tensor

$$R^{\alpha\beta} = \frac{\partial x^\alpha}{\partial x^{\hat{\mu}}}\frac{\partial x^\beta}{\partial x^{\hat{\nu}}} R^{\hat{\mu}\hat{\nu}} \tag{22}$$

contains all the coordinate specifics:

$$R^{\alpha\beta} = \begin{pmatrix} E & F^R & R^{-1}F^\theta & F^z \\ F^R & P^{RR} & R^{-1}P^{R\theta} & P^{Rz} \\ R^{-1}F^\theta & R^{-1}P^{R\theta} & R^{-2}P^{\theta\theta} & R^{-1}P^{\theta z} \\ F^z & P^{Rz} & R^{-1}P^{Rz} & P^{zz} \end{pmatrix}. \tag{23}$$

Applying the Lorentz transformation between the fixed- and comoving-frame moments yields the transformation law between them [13,14]:

$$\begin{aligned}
E_{co} &= \gamma^2\left[E - 2v_i F^i + v_i v_j P^{ij}\right], \\
F^i_{co} &= \left[\delta^i_j + (\frac{\gamma-1}{v^2} + \gamma^2) v^i v_j\right] F^j - \gamma^2 v^i E \\
&\quad - \gamma v_j \left[\delta^i_k + \frac{\gamma-1}{v^2} v^i v_k\right] P^{jk}, \\
P^{ij}_{co} &= \gamma^2 v^i v^j E - \gamma\left[v^i \delta^j_k + v^j \delta^i_k + 2\frac{\gamma-1}{v^2} v^i v^j v_k\right] F^k \\
&\quad + (\delta^i_k + \frac{\gamma-1}{v^2} v^i v_k)(\delta^j_l + \frac{\gamma-1}{v^2} v^j v_l) P^{kl}.
\end{aligned} \tag{24}$$

3. Radiation Four-force Density

Since the interactions between radiation and matter are most naturally described in the comoving frame, first the comoving-frame tetrad components of the radiation four-force density are defined as

$$G^{\hat{\alpha}}_{co} = \frac{1}{c}\int d\nu_{co} \int d\Omega_{co} [\chi_{co} I_{\nu_{co}} - \eta_{co}] n^{\hat{\alpha}}_{co}. \tag{25}$$

In the case of isotropic scattering and emission, $G^{\hat{\alpha}}_{co}$ can be written in terms of more familiar quantities, the heating and cooling functions, Γ_{co} and Λ_{co}, and the opacity, $\bar{\chi}_{co}$, all defined in the comoving frame:

$$G^{\hat{t}}_{co} = \Gamma_{co} - \Lambda_{co}, \quad G^{\hat{i}}_{co} = \bar{\chi}_{co} F^{i}_{co}, \tag{26}$$

where

$$\begin{aligned}\Gamma_{co} &\equiv \frac{1}{c}\int d\nu_{co}\int d\Omega_{co}\chi_{co}I_{\nu_{co}},\\ \Lambda_{co} &\equiv \frac{1}{c}\int d\nu_{co}\int d\Omega_{co}\eta_{co},\\ \bar{\chi}_{co}F^{i}_{co} &\equiv \frac{1}{c}\int d\nu_{co}\int d\Omega_{co}\chi_{co}I_{\nu_{co}}n^{i}_{co}.\end{aligned} \tag{27}$$

The contravariant components of the radiation four-force density are related to the tetrad components by

$$\begin{aligned}G^t &= \gamma G^{\hat{t}}_{co} + \gamma v^i G^{\hat{i}}_{co},\\ G^R &= G^{\hat{R}}_{co} + \gamma v^R G^{\hat{t}}_{co} + \frac{\gamma-1}{v^2}v^R v_i G^{\hat{i}}_{co},\\ RG^\theta &= G^{\hat{\theta}}_{co} + \gamma v^\theta G^{\hat{t}}_{co} + \frac{\gamma-1}{v^2}v^\theta v_i G^{\hat{i}}_{co},\\ G^z &= G^{\hat{z}}_{co} + \gamma v^z G^{\hat{t}}_{co} + \frac{\gamma-1}{v^2}v^z v_i G^{\hat{i}}_{co}.\end{aligned} \tag{28}$$

IV. RADIATION HYDRODYNAMIC EQUATIONS

1. Continuity Equation

Now, the continuity equation, Eq. (3), in cylindrical coordinates becomes

$$\frac{\partial}{\partial t}(\gamma n)+\frac{1}{R}\frac{\partial}{\partial R}(RnU^R)+\frac{\partial}{\partial\theta}(nU^\theta)+\frac{\partial}{\partial z}(nU^z) = 0. \tag{29}$$

2. Euler and Energy Equations

Projecting Eq. (6) perpendicular to U^α yields the relativistic Euler equations, the R-component of which is

$$\begin{aligned}&\gamma\omega_g\frac{\partial U^R}{\partial t} + \omega_g U^i\frac{\partial U^R}{\partial x^i} - \omega_g R(U^\theta)^2\\ &+\frac{\partial P_g}{\partial R} + \gamma U^R\frac{\partial P_g}{\partial t} + U^R U^i\frac{\partial P_g}{\partial x^i}\\ &= -\gamma U^R G^t + [1+(U^R)^2]G^R + R^2U^RU^\theta G^\theta\\ &+U^RU^zG^z + f^R + U^RU_\beta f^\beta,\end{aligned} \tag{30}$$

while the θ-component of the equation is

$$\begin{aligned}&\gamma\omega_g\frac{\partial U^\theta}{\partial t} + \omega_g U^i\frac{\partial U^\theta}{\partial x^i} + 2\omega_g\frac{U^RU^\theta}{R}\\ &+\frac{1}{R^2}\frac{\partial P_g}{\partial \theta} + \gamma U^\theta\frac{\partial P_g}{\partial t} + U^\theta U^i\frac{\partial P_g}{\partial x^i}\\ &= -\gamma U^\theta G^t + U^\theta U^R G^R + [1+R^2(U^\theta)^2]G^\theta\\ &+U^\theta U^zG^z + f^\theta + U^\theta U_\beta f^\beta,\end{aligned} \tag{31}$$

and the z-component of the equation is

$$\begin{aligned}&\gamma\omega_g\frac{\partial U^z}{\partial t} + \omega_g U^i\frac{\partial U^z}{\partial x^i}\\ &+\frac{\partial P_g}{\partial z} + \gamma U^z\frac{\partial P_g}{\partial t} + U^z U^i\frac{\partial P_g}{\partial x^i}\\ &= -\gamma U^z G^t + U^z U^R G^R + R^2U^zU^\theta G^\theta\\ &+[1+(U^z)^2]G^z + f^z + U^zU_\beta f^\beta.\end{aligned} \tag{32}$$

The energy equation for the matter is obtained by projecting Eq. (6) along U^α:

$$\begin{aligned}&-nU^t\frac{\partial}{\partial t}\left(\frac{\omega_g}{n}\right) - nU^i\frac{\partial}{\partial x^i}\left(\frac{\omega_g}{n}\right) + U^t\frac{\partial P_g}{\partial t} + U^i\frac{\partial P_g}{\partial x^i}\\ &= -G^{\hat{t}}_{co} + U_\alpha f^\alpha = \Lambda_{co} - \Gamma_{co} + U_\alpha f^\alpha.\end{aligned} \tag{33}$$

3. Radiation Moment Equations

The radiation energy equation is the time component of Eq. (7),

$$\begin{aligned}&\frac{\partial E}{\partial t} + \frac{1}{R}\frac{\partial}{\partial R}(RF^R) + \frac{1}{R}\frac{\partial F^\theta}{\partial\theta} + \frac{\partial F^z}{\partial z}\\ &= -G^t = \gamma\big(\Lambda_{co} - \Gamma_{co}\big) - \bar{\chi}_{co}\tilde{u}_i F^i_{co}.\end{aligned} \tag{34}$$

The left-hand side of Eq. (34) appears simpler than the corresponding equation in comoving coordinates (see e.g., Refs. [4] and [15]) because the temporal and the spatial derivatives of the fixed-frame moments are calculated with respect to the fixed coordinates while the interactions between matter and radiation are described by the comoving-frame quantities. This 'mixed-frame' description simplifies the equation and makes each term easier to understand.

The R-component of Eq. (7) produces the radiation momentum equation in the R-direction:

$$\begin{aligned}&\frac{\partial F^R}{\partial t} + \frac{\partial P^{RR}}{\partial R} + \frac{1}{R}\frac{\partial P^{R\theta}}{\partial\theta} + \frac{\partial P^{Rz}}{\partial z} + \frac{P^{RR}-P^{\theta\theta}}{R}\\ &= -\bar{\chi}_{co}F^R_{co} - \tilde{u}^R(\Gamma_{co}-\Lambda_{co}) - \frac{\gamma-1}{\tilde{u}^2}\tilde{u}^R\tilde{u}_i\bar{\chi}_{co}F^i_{co}.\end{aligned} \tag{35}$$

The radiation momentum equation in the θ-direction from Eq. (7) is

$$\begin{aligned}&\frac{\partial F^\theta}{\partial t} + \frac{\partial P^{R\theta}}{\partial R} + \frac{1}{R}\frac{\partial P^{\theta\theta}}{\partial\theta} + \frac{\partial P^{\theta z}}{\partial z} + \frac{2P^{R\theta}}{R}\\ &= -\bar{\chi}_{co}F^\theta_{co} - \tilde{u}^\theta(\Gamma_{co}-\Lambda_{co}) - \frac{\gamma-1}{\tilde{u}^2}\tilde{u}^\theta\tilde{u}_i\bar{\chi}_{co}F^i_{co},\end{aligned} \tag{36}$$

and that in the z-direction is

$$\begin{aligned}&\frac{\partial F^z}{\partial t} + \frac{\partial P^{Rz}}{\partial R} + \frac{1}{R}\frac{\partial P^{\theta z}}{\partial\theta} + \frac{\partial P^{zz}}{\partial z} + \frac{P^{Rz}}{R}\\ &= -\bar{\chi}_{co}F^z_{co} - \tilde{u}^z(\Gamma_{co}-\Lambda_{co}) - \frac{\gamma-1}{\tilde{u}^2}\tilde{u}^z\tilde{u}_i\bar{\chi}_{co}F^i_{co}.\end{aligned} \tag{37}$$

4. Closure Relation

Due to the innate nature of the moment expansion, moment equations do not constitute a complete system of equations: the number of unknown variables is always larger than that of the equations. For example, we need 16 variables to describe the matter and the radiation field: n, T, U^α, E, F^i, and P^{ij}. The number of equations, however, is only 10: 1 from the normalization of U^α (Eq. (10)), 1 from the continuity equation (Eq. (29)), 3 from the Euler equations (Eqs. (30)–(32)), 1 from the energy equation (Eq. (33)), and 4 from the radiation moment equations (Eqs. (35)–(37)). Thus we need to provide additional relations to close the system of equations. The most popular closure relation for a one-dimensional problem is the Eddington factor that relates the second radiation moment P to the zeroth radiation moment E,

$$f_E \equiv \frac{P}{E}, \tag{38}$$

which has an asymptotic value 1/3 when $\tau \gg 1$ and 1 when $\tau \ll 1$. The exact shape of f_E can be calculated from fully angle-dependent radiative transfer calculations, generally for a static or steady one-dimensional flow with a prescribed velocity structure [16–18]. Sometimes, for a complex dynamic radiation flow an educated guess of f_E as a function of the optical depth is the best we can do [19,20]. However, it is not proven at all that such a treatment would be valid for a flow with a strong velocity gradient or for a three-dimensional flow.

Since there are virtually no fully angle- and time-dependent radiative transfer calculations for a three-dimensional relativistic flow, the detailed form of f_E for such a problem is entirely unknown. We may utilize a generalized form of the Eddington factor derived from the maximum entropy method by Minerbo [21], but little is known about its validity or accuracy in specific problems, except for steady and symmetric cases [22]. Therefore, until we accumulate more knowledge about the relativistic three-dimensional radiative transfer, we need to be cautious about applying the radiation moment formalism with a prescribed Eddington factor.

Nonetheless, there are cases in which the radiation field, or equally, radiation moments, are trivially determined. I will consider one such example.

V. EXAMPLE: SPHERICALLY SYMMETRIC RADIATION FIELD

Certain classes of astrophysical accretion systems can be approximated by a cylindrical gas flow with a point-like central radiation source. I will apply the radiation hydrodynamic equations to such cases to derive the equation of motion for the gas. Although the current treatment is correct up to arbitrary order in $\boldsymbol{v}$, I will keep only terms up to the first order in $\boldsymbol{v}$ for clarity.

When the central radiation source is at rest with respect to the coordinates and photons stream out radially, the radiation flux measured by a fixed-frame observer is simply

$$\boldsymbol{F} = F^r \hat{r} = F^R \hat{R} + F^z \hat{z}, \tag{39}$$

where

$$F^r \equiv \frac{L}{4\pi r^2}, \quad F^R \equiv F^r \left(\frac{R}{r}\right), \quad F^z \equiv F^r \left(\frac{z}{r}\right), \tag{40}$$

and E denotes the radiation energy density measured by a fixed observer. In general, $E \neq F^r$, but in the streaming limit, $E \approx F^r$. From Eq. (24), the comoving flux measured by an observer comoving with the gas becomes

$$\begin{aligned} F^R_{co} &= F^R - \left[v^R(1+\frac{R^2}{r^2}) + v^z\frac{Rz}{r^2}\right]E, \\ F^\theta_{co} &= -v^\theta E, \\ F^z_{co} &= F^z - \left[v^z(1+\frac{z^2}{r^2}) + v^R\frac{Rz}{r^2}\right]E. \end{aligned} \tag{41}$$

The tetrad form of the radiation stress energy tensor is

$$R^{\hat\alpha\hat\beta} = \begin{pmatrix} E & F^R & 0 & F^z \\ F^R & \frac{R^2}{r^2}E & 0 & \frac{Rz}{r^2}E \\ 0 & 0 & 0 & 0 \\ F^z & \frac{Rz}{r^2}E & 0 & \frac{z^2}{r^2}E \end{pmatrix}. \tag{42}$$

Also, to the first order in $\boldsymbol{v}$, $\gamma = 1 + O(v^2)$, $\tilde{u}^i = v^i + O(v^2)$, and $\bar\chi_{co} = \bar\chi + O(v^2)$.

Substituting the above definitions and simplification to Eqs. (30)–(32) yields the equations of motion. The R-component of the equation of motion is

$$\begin{aligned} &\omega_g\frac{\partial v^R}{\partial t} + \omega_g v^i\frac{\partial v^R}{\partial x^i} - \omega_g R(v^\theta)^2 \\ &+\frac{\partial P_g}{\partial R} + v^R\frac{\partial P_g}{\partial t} + v^R v^i \frac{\partial P_g}{\partial x^i} \\ &= f^R + \bar\chi F^R - \left[v^R(1+\frac{R^2}{r^2}) + v^z\frac{Rz}{r^2}\right]\bar\chi E + O(v^2). \end{aligned} \tag{43}$$

The third term on the right-hand side of the equation acts as a radiation drag and reflects the fact that the moving gas sees the comoving flux F^R_{co} rather than the fixed-frame flux F^R. The θ-component of the equation of motion is

$$\begin{aligned} &\omega_g\frac{\partial v^\theta}{\partial t} + \omega_g v^i\frac{\partial v^\theta}{\partial x^i} + 2\omega_g\frac{v^R v^\theta}{R} \\ &+\frac{1}{R}\frac{\partial P_g}{\partial \theta} + v^\theta\frac{\partial P_g}{\partial t} + v^\theta v^i \frac{\partial P_g}{\partial x^i} \\ &= f^\theta - \bar\chi v^\theta E + O(v^2). \end{aligned} \tag{44}$$

The second term on the right-hand side, $-\bar\chi v^\theta E$, is the radiation drag in the θ-direction due to an aberration of the radially streaming photons, namely, the Poynting-Robertson effect. The z-component of the equation is

$$\omega_g\frac{\partial v^z}{\partial t} + \omega_g v^i\frac{\partial v^z}{\partial x^i}$$

$$+\frac{\partial P_g}{\partial z} + v^z \frac{\partial P_g}{\partial t} + v^z v^i \frac{\partial P_g}{\partial x^i} \tag{45}$$
$$= f^z + \bar{\chi} F^z - \left[v^R \frac{Rz}{r^2} + v^z (1 + \frac{z^2}{r^2}) \right] \bar{\chi} E + O(v^2),$$

and we again see a radiation drag in z-direction due to an aberration of photons. Finally, the gas energy equation is of the same form as in the spherical coordinates [7]:

$$-n\frac{\partial}{\partial t}\left(\frac{\omega_g}{n}\right) - nv^i \frac{\partial}{\partial x^i}\left(\frac{\omega_g}{n}\right) + \frac{\partial P_g}{\partial t} + v^i \frac{\partial P_g}{\partial x^i}$$
$$= \Lambda_{co} - \Gamma_{co} - f^t + v_i f^i. \tag{46}$$

VI. SUMMARY

Three-dimensional special-relativistic, hydrodynamic and radiation moment equations are derived. Matter and radiation quantities are defined in fixed and comoving tetrads and are transformed to corresponding covariant forms. The interactions between radiation and matter are described similarly by the radiation four-force density defined in the comoving tetrad and transformed to the covariant form. Conservation of energy and momentum tensor for matter and, separately, for radiation leads to the relativistic hydrodynamic and radiation moment equations. The current approach is a mixed-frame formalism in the sense that the physical quantities and interactions are described in the comoving frame while the physics of matter and radiation is described within the fixed coordinate system. This formalism is general and can be applied to any coordinates or spacetimes, but in this paper, I specifically showed how to derive the equations in cylindrical coordinates. As an example, I also applied the formalism to a cylindrical gas flow with a central light source to obtain the relativistic equations of motion relevant for accretion disks and jets.

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