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A Classification of Plane Symmetric Kinematic Self-Similar Solutions

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In this paper, we provide a classification of plane symmetric kinematic self-similar perfect-fluid and dust solutions. In the perfect-fluid and dust cases, kinematic self-similar vectors of the first, second, zeroth, and infinite kinds for the tilted, orthogonal, and parallel cases have been explored with different equations of state. We obtain a total of eleven plane symmetric kinematic self-similar solutions out of which six are independent. The perfect-fluid case gives two solutions: infinite tilted and infinite orthogonal kinds of self-similarity. In the dust case, we have four independent solutions: first orthogonal, infinite tilted, infinite orthogonal, and infinite parallel kinds of self-similarity. The remaining cases are not consistent. It is interesting to mention that some of these solutions turn out to be vacuum.

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I. INTRODUCTION

The General Theory of Relativity (GR), which is a field theory of gravitation and is described in terms of geometry, is highly non-linear. Because of this non-linearity, it becomes very difficult to solve the gravitational field equations unless certain symmetry restrictions are imposed on the spacetime metric. These symmetry restrictions are expressed in terms of isometries possessed by spacetimes. These isometries, which are also called Killing Vectors (KVs), give rise to conservation laws [1].

There has been recent literature [2–8] and references therein, which shows a significant interest in the study of various symmetries. These symmetries arise in the exact solutions of the Einstein field equations (EFEs) given by

$$G_{ab} = R_{ab} - \frac{1}{2}Rg_{ab} = \kappa T_{ab}, \tag{1}$$

where G_{ab} represents the components of the Einstein tensor, R_{ab} are the components of the Ricci tensor, and T_{ab} are the components of the matter (or energy-momentum) tensor, R is the Ricci scalar, and κ is the gravitational constant. The geometrical nature of a spacetime is expressed by the metric tensor through the EFEs.

Self-similarity is very helpful in simplifying the field equations. Self-similarity leads to a reduction of the governing equations from partial differential equations to ordinary differential equations, whose mathematical treatment is relatively simple. Although self-similar solutions are only special solutions, they often play an important role as an intermediate attractor. There does not exist any characteristic scale in Newtonian gravity or GR. Invariance of the field equations under a scale transformation indicates that there exist scale invariant solutions to the EFEs. These solutions are known as self-similar solutions.

In order to obtain realistic solutions of gravitational collapse leading to star formation, many authors have investigated self-similar solutions in Newtonian gravity [9]. There exist several preferred geometric structures in self-similar models, and a number of natural approaches may be used in studying them. The three most common ones are the co-moving, the homothetic, and the Schwarzschild approaches. In this paper, we shall use the co-moving approach. In the co-moving approach, pioneered by Cahill and Taub [10], the coordinates are adopted to the fluid 4-velocity vector. This probably affords the best physical insight and is the most convenient one. In GR, self-similarity is defined by the existence of a homothetic vector (HV) field. Such similarity is called the first kind (or homothety or continuous self similarity (CSS)). There exists a natural generalization of homothety called kinematic self-similarity, which is defined by the existence of a kinematic self-similar (KSS) vector field. Kinematic self-similarity is characterized by an index α/δ (similarity index) and can be classified into three kinds. The basic condition characterizing a manifold vector field ξ as a self-similar generator is given by

$$\pounds_\xi A = \lambda A, \tag{2}$$

where λ is a constant, A is an independent physical field, and $\pounds_\xi$ denotes the Lie derivative along ξ. This field can be scalar (*e.g.*, μ), vector (*e.g.*, u_a), or tensor (*e.g.*, g_{ab}).

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In GR, the gravitational field is represented by the metric tensor g_{ab}, and an appropriate definition of geometrical self-similarity is necessary.

The vector field ξ can have three cases, *i.e.*, parallel, orthogonal, and tilted. They are distinguished by the relation between the generator and a timelike vector field, which is identified as the fluid flow, if it exists. The tilted case is the most general among them.

The self-similar idea of Cahil and Taub [10] corresponds to Newtonian self-similarity of the homothetic class. Carter and Henriksen [11, 12] defined the other kinds of self-similarity, namely, the second, zeroth, and infinite kinds. In the context of kinematic self-similarity, homothety is considered as the first kind. Several authors have explored KSS perfect fluid solutions. The only compatible barotropic equation of state with self-similarity of the first kind is

$$p = k\rho. \tag{3}$$

Carr [2] has classified the self-similar perfect-fluid solutions of the first kind in the dust case ($k = 0$). The case $0 < k < 1$ has been studied by Carr and Coley [3]. Coley [13] has shown that the Friedmann-Robertson-Walker solution is the only spherically symmetric homothetic perfect fluid solution in the parallel case. McIntosh [14] has discussed the fact that a stiff fluid ($k = 1$) is the only compatible perfect fluid with homothety in the orthogonal case. Benoit and Coley [15] have studied analytic spherically symmetric solutions of the EFEs coupled with a perfect fluid and admitting a KSS vector of the first, second, and zeroth kinds.

Carr *et al.* [16] have considered the KSS associated with the critical behavior observed in the gravitational collapse of spherically symmetric perfect-fluid with an equation of state $p = k\rho$. They showed for the first time the global nature of these solutions and showed that it is sensitive to the value of α. Carr *et al.* [17], further, investigated the solution space of self-similar spherically symmetric perfect-fluid models and the physical aspects of those solutions. They combined the state space description of the homothetic approach with the use of the physically interesting quantities arising in the comoving approach. Coley and Goliath [18] investigated self-similar spherically symmetric cosmological models with a perfect-fluid and a scalar field with an exponential potential.

Gravitational collapse is one of the fundamental problems in GR. Self-similar gravitational collapse and critical collapse provide information about the collapse of an object. The collapse generally has three kinds of possible final states. First is simply the halt of the processes in a self-sustained object or the description of a matter field or a gravitational field. The second is the formation of black holes with outgoing gravitational radiation and matter while the third is the formation of naked singularities. Critical collapse in the context of self-similarity gives information about the mass of a black hole formed as a result of a collapse.

Recently, Maeda *et al.* [4, 5] studied the KSS vector of the second, zeroth, and infinite kinds in the tilted case. They assumed a perfect-fluid spacetime obeying a relativistic polytropic equations of state. Further, they assumed two kinds of polytropic equations of state and showed that such spacetimes must be vacuum in both cases. They explored the case in which a KSS vector is not only tilted to the fluid flow but also parallel or orthogonal. The same authors [6] also discussed the classification of the spherically symmetric KSS perfect-fluid and dust solutions. That analysis provided some interesting solutions.

In a recent paper, Sharif and Sehar [8] investigated the KSS solutions for cylindrically symmetric spacetimes. An analysis was given for the perfect-fluid and dust cases with tilted, parallel, and orthogonal vectors by using different equations of state. Some interesting consequences developed. The same authors also studied the properties of such solutions for spherically symmetric [19], cylindrically symmetric [20], and plane symmetric spacetimes [21].

The group G_3 contains two special cases of particular physical interest: spherical and plane symmetry. In this paper, we shall use the same procedure to investigate KSS solutions for plane symmetric spacetimes. The paper has been organised as follows. In Section II, we shall discuss KSS vectors of different kinds for plane symmetric spacetimes. Section III is devoted to the titled perfect-fluid case. In Section IV, we shall find the titled dust solutions. Sections V and VI are used to explore the orthogonal perfect-fluid and dust solutions, respectively. Sections VII and VIII are devoted to the parallel perfect-fluid and dust cases. Finally, we shall summarize and discuss all the results.

II. PLANE SYMMETRIC SPACETIME AND KINEMATIC SELF-SIMILARITY

A plane symmetric Lorentzian manifold is defined to be a manifold that admits the group $SO(2) \times \Re^2$ as the minimal isometry group in such a way that the group orbits are spacelike surfaces of zero curvature, where $SO(2)$ corresponds to a rotation and $\Re^2$ to translations along the spatial directions y and z. The metric for the most general plane symmetric spacetime has the following form [22]:

$$ds^2 = e^{2\nu(t,x)}dt^2 - e^{2\mu(t,x)}dx^2 - e^{2\lambda(t,x)}(dy^2 + dz^2), \tag{4}$$

where ν, μ, and λ are arbitrary functions of t and x. The metric has three isometries given as $\xi_1 = \partial_x$, $\xi_2 = \partial_y$, $\xi_3 = x\partial_y - y\partial_x$. This metric can further be classified according to the additional isometries it admits. For the sake of simplicity, we take the coefficient of dx^2 as unity. The corresponding metric reduces to

$$ds^2 = e^{2\nu(t,x)}dt^2 - dx^2 - e^{2\lambda(t,x)}(dy^2 + dz^2). \tag{5}$$

The energy-momentum tensor for a perfect fluid can be written as

$$T_{ab} = [\rho(t,x) + p(t,x)]u_a u_b - p(t,x)g_{ab}, \quad (a,b = 0,1,2,3), \tag{6}$$

where ρ is the density, p is the pressure, and u_a is the four-velocity of the fluid element. In the co-moving coordinate system, the four-velocity can be written as $u_a = (e^{\nu(t,x)}, 0, 0, 0)$. The EFEs become

$$\kappa\rho = e^{-2\nu}{\lambda_t}^2 - 3{\lambda_x}^2 - 2\lambda_{xx}, \tag{7}$$

$$0 = \lambda_{tx} - \lambda_t\nu_x + \lambda_t\lambda_x, \tag{8}$$

$$\kappa p = {\lambda_x}^2 + 2\lambda_x\nu_x - e^{-2\nu}(2\lambda_{tt} - 2\lambda_t\nu_t + 3{\lambda_t}^2), \tag{9}$$

$$\kappa p = \nu_{xx} + {\nu_x}^2 + \nu_x\lambda_x + {\lambda_x}^2 + \lambda_{xx} - e^{-2\nu}(\lambda_{tt} - \lambda_t\nu_t + {\lambda_t}^2). \tag{10}$$

Conservation of energy-momentum tensor, ${T^{ab}}_{;b} = 0$, provides the following two equations:

$$\lambda_t = -\frac{\rho_t}{2(\rho + p)}, \tag{11}$$

and

$$\nu_x = \frac{p_x}{(\rho + p)}. \tag{12}$$

The general form of a vector field ξ, for a plane symmetric spacetime, can take the following form:

$$\xi^a\frac{\partial}{\partial x^a} = h_1(t,x)\frac{\partial}{\partial t} + h_2(t,x)\frac{\partial}{\partial x}, \tag{13}$$

where h_1 and h_2 are arbitrary functions. When ξ is parallel to the fluid flow, $h_2 = 0$, and when ξ is orthogonal to the fluid flow, $h_1 = 0$. When both h_1 and h_2 are non-zero, ξ is tilted to the fluid flow.

A KSS vector ξ satisfies the following conditions:

$$\pounds_\xi h_{ab} = 2\delta h_{ab}, \tag{14}$$

$$\pounds_\xi u_a = \alpha u_a, \tag{15}$$

where $h_{ab} = g_{ab} - u_a u_b$ is the projection tensor, and α and δ are constants. The similarity transformation is characterized by the scale independent ratio α/δ. This ratio is referred to as the similarity index and yields the following two cases according to

1. $\delta \neq 0$,
2. $\delta = 0$.

Case 1: If $\delta \neq 0$, it can be chosen as unity, and the KSS vector for the titled case can take the following form:

$$\xi^a\frac{\partial}{\partial x^a} = (\alpha t + \beta)\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}. \tag{16}$$

For this case, the similarity index, α/δ, further yields the following three different possibilities:

(i) $\delta \neq 0$, $\alpha = 1$ (*β can be taken to be zero*),
(ii) $\delta \neq 0$, $\alpha = 0$ (*β can be taken to be unity*),
(iii) $\delta \neq 0$, $\alpha \neq 0, 1$ (*β can be taken to be zero*).

Case 1(i) corresponds to a self-similarity of the *first kind*. In this case, ξ is a homothetic vector, and the self-similar variable ξ turns out to be x/t. For the case 1(ii), it corresponds to a self-similarity of the *zeroth kind*, and the self-similar variable takes the form

$$\xi = xe^{-t}.$$

Case 1(iii) corresponds to a self-similarity of the *second kind*, and the self-similar variable becomes

$$\xi = \frac{x}{(\alpha t)^{\frac{1}{\alpha}}}.$$

It turns out that for case (1), when $\delta \neq 0$, with a self-similar variable ξ, the metric functions become

$$\nu = \nu(\xi), \quad e^\lambda = xe^{\lambda(\xi)}. \tag{17}$$

For case (2), in which $\delta = 0$ and $\alpha \neq 0$ (α can be unity and β can be re-scaled to zero), the self-similarity is known as an *infinite kind*, and the KSS vector ξ turns out to be

$$\xi^a\frac{\partial}{\partial x^a} = t\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}, \tag{18}$$

and the self-similar variable will become

$$\xi = e^{\frac{x}{c}}/t,$$

where c is an arbitrary constant. The metric functions will be of the form

$$\nu = \nu(\xi), \quad \lambda = \lambda(\xi). \tag{19}$$

Notice that for a plane symmetric spacetime, the self-similar variables of the first, second, and zeroth kinds turn out to be the same as for the spherically and cylindrically symmetric spacetimes with the exception that r has been replaced by x in the plane symmetric metric. We note that for $\delta = 0 = \alpha$, the KSS vector ξ becomes KV.

If the KSS vector ξ is parallel to the fluid flow, it follows that

$$\xi^a\frac{\partial}{\partial x^a} = f(t)\frac{\partial}{\partial t}, \tag{20}$$

where $f(t)$ is an arbitrary function. It is mentioned here that the self-similar variable for the spherically symmetric metric is r whereas it turns out to be r only for a self-similarity of the infinite kind in the case of a cylindrically symmetric metric. In the remaining kinds of self-similarity, we obtain contradictory results for the cylindrically symmetric spacetime. For the plane symmetry, we obtain contradictory results for the first, second, and zeroth kinds of KSS while for the infinite kind, the self-similar variable turns out to be x. This implies that no solution exists when ξ is parallel to the fluid flow for the first, second, and zeroth kinds of KSS, but some solutions may exist in the case of an infinite kind of self-similarity.

When the KSS vector ξ is orthogonal to the fluid flow, we obtain

$$\xi^a \frac{\partial}{\partial x^a} = g(x)\frac{\partial}{\partial x}, \tag{21}$$

where $g(x)$ is an arbitrary function and the self-similar variable is t.

We assume the following two types of polytropic equation of states (EOS). We denote the first equation of state by EOS(1):

$$p = k\rho^{\gamma},$$

where k and γ are constants. The other EOS can be written as [17]

$$p = kn^{\gamma},$$
$$\rho = m_b n + \frac{p}{\gamma - 1},$$

where m_b is a constant and corresponds to the baryon mass, and $n(t,r)$ corresponds to baryon number density. This equation is called the second equation of state and is written as EOS(2). For EOS(1) and EOS(2), we take $k \neq 0$ and $\gamma \neq 0, 1$. A third equation of state, denoted by EOS(3), is the following:

$$p = k\rho.$$

Here, we assume that $-1 \leq k \leq 1$ and $k \neq 0$.

For different values of γ, EOS(1) and EOS(2) have different properties. A thermodynamic instability of the fluid is seen for $\gamma < 0$. For $0 < \gamma < 1$, both EOS(1) and EOS(2) are approximated by a dust fluid in the high-density region. For $\gamma > 1$, EOS(2) is approximated by EOS(3) with $k = \gamma - 1$ in the high-density region. The cases $\gamma > 2$ for EOS(2) and $\gamma > 1$ for EOS(2) show that the dominant energy condition can be violated in the high-density region, which is physically not interesting [5].

III. TILTED PERFECT FLUID CASE

1. Self-similarity of the First Kind

Firstly, we discuss the self-similarity of the first kind for the tilted perfect-fluid case. In this case, it follows from the EFEs that the energy density ρ and the pressure p must take the forms

$$\kappa\rho = \frac{1}{x^2}[\rho_1(\xi) + \frac{x^2}{t^2}\rho_2(\xi)], \tag{22}$$
$$\kappa p = \frac{1}{x^2}[p_1(\xi) + \frac{x^2}{t^2}p_2(\xi)], \tag{23}$$

where the self-similar variable is $\xi = x/t$. If the EFEs and the equations of motion for the matter field are satisfied for the $O[(\frac{x}{t})^0]$ and the $O[(\frac{x}{t})^2]$ terms separately, we obtain a set of ordinary differential equations. Thus, Eqs. (7)-(12) reduce to the following:

$$-\dot{\rho_1} = 2\dot{\lambda}(\rho_1 + p_1), \tag{24}$$
$$\dot{\rho_2} + 2\rho_2 = -2\dot{\lambda}(\rho_2 + p_2), \tag{25}$$
$$\dot{p_1} - 2p_1 = \dot{\nu}(\rho_1 + p_1), \tag{26}$$
$$\dot{p_2} = \dot{\nu}(\rho_2 + p_2), \tag{27}$$
$$-\rho_1 = 1 + 4\dot{\lambda} + 3\dot{\lambda}^2 + 2\ddot{\lambda}, \tag{28}$$
$$\rho_2 = \dot{\lambda}^2 e^{-2\nu}, \tag{29}$$
$$0 = \ddot{\lambda} + \dot{\lambda}^2 + \dot{\lambda} - \dot{\lambda}\dot{\nu}, \tag{30}$$
$$p_1 = 1 + 2\dot{\lambda} + \dot{\lambda}^2 + 2\dot{\nu} + 2\dot{\lambda}\dot{\nu}, \tag{31}$$
$$-e^{2\nu}p_2 = 2\ddot{\lambda} + 3\dot{\lambda}^2 + 2\dot{\lambda} - 2\dot{\lambda}\dot{\nu}, \tag{32}$$
$$p_1 = \ddot{\lambda} + \dot{\lambda}^2 + \dot{\lambda} + \dot{\lambda}\dot{\nu} + \ddot{\nu} + \dot{\nu}^2, \tag{33}$$
$$-e^{2\nu}p_2 = \ddot{\lambda} + \dot{\lambda}^2 + \dot{\lambda} - \dot{\lambda}\dot{\nu}, \tag{34}$$

where dot (.) represents a derivative with respect to $ln(\xi)$. When we use Eq. (30) in Eq. (34), it turns out that $p_2 = 0$, which together with Eqs. (30) and (32) yields λ as an arbitrary constant. Solving the above equations simultaneously, we get $\rho_1 = -1$ from Eq. (28) and $\rho_2 = 0$ from Eq. (29). Finally, we are left with three equations in two unknowns p_1 and ν. When we solve these equations simultaneously, we do not have values of p_1 and ν that satisfy the three equations. Hence, there is no solution in this case.

2. Self-similarity of the Second Kind

Here, we discuss the self-similarity of the second kind for the tilted perfect-fluid case. In this case, it follows from the EFEs that the energy density ρ and the pressure p must take the forms

$$\kappa\rho = \frac{1}{x^2}[\rho_1(\xi) + \frac{x^2}{t^2}\rho_2(\xi)], \tag{35}$$
$$\kappa p = \frac{1}{x^2}[p_1(\xi) + \frac{x^2}{t^2}p_2(\xi)], \tag{36}$$

where the self-similar variable is $\xi = x/(\alpha t)^{\frac{1}{\alpha}}$. If the EFEs and the equations of motion for the matter field are satisfied for the $O[(\frac{x}{t})^0]$ and the $O[(\frac{x}{t})^2]$ terms separately, we obtain a set of ordinary differential equations. Thus, Eqs. (7)-(12) take the following forms:

$$\dot{\rho_1} = -2\dot{\lambda}(\rho_1 + p_1), \tag{37}$$
$$\dot{\rho_2} + 2\alpha\rho_2 = -2\dot{\lambda}(\rho_2 + p_2), \tag{38}$$
$$\dot{p_1} - 2p_1 = \dot{\nu}(\rho_1 + p_1), \tag{39}$$
$$\dot{p_2} = \dot{\nu}(\rho_2 + p_2), \tag{40}$$
$$0 = \ddot{\lambda} + \dot{\lambda}^2 + \dot{\lambda} - \dot{\lambda}\dot{\nu}, \tag{41}$$
$$-\rho_1 = 1 + 4\dot{\lambda} + 3\dot{\lambda}^2 + 2\ddot{\lambda}, \tag{42}$$
$$\alpha^2\rho_2 = \dot{\lambda}^2 e^{-2\nu}, \tag{43}$$

$$p_1 = 1 + 2\dot{\lambda} + \dot{\lambda}^2 + 2\dot{\nu} + 2\dot{\lambda}\dot{\nu}, \tag{44}$$

$$-\alpha^2 e^{2\nu} p_2 = 2\ddot{\lambda} + 3\dot{\lambda}^2 + 2\alpha\dot{\lambda} - 2\dot{\lambda}\dot{\nu}, \tag{45}$$

$$p_1 = \ddot{\lambda} + \dot{\lambda}^2 + \dot{\lambda} + \dot{\lambda}\dot{\nu} + \ddot{\nu} + \dot{\nu}^2, \tag{46}$$

$$-\alpha^2 e^{2\nu} p_2 = \ddot{\lambda} + \dot{\lambda}^2 + \alpha\dot{\lambda} - \dot{\lambda}\dot{\nu}. \tag{47}$$

Now, we solve this set of equations by using EOS (1)-(3).

A. Equations of State (1) and (2)

If a perfect-fluid satisfies EOS(1) for $k \neq 0$ and $\gamma \neq 0, 1$, Eqs. (35) and (36) become

$$\alpha = \gamma, \quad p_1 = 0 = \rho_2, \quad p_2 = \frac{k}{(8\pi G)^{(\gamma-1)}\gamma^2}\xi^{-2\gamma}{\rho_1}^{\gamma}, \quad [Case\ I] \tag{48}$$

or

$$\alpha = \frac{1}{\gamma}, \quad p_2 = 0 = \rho_1, \quad p_1 = \frac{k}{(8\pi G)^{(\gamma-1)}\gamma^{2\gamma}}\xi^2{\rho_2}^{\gamma} \quad [Case\ II]. \tag{49}$$

If a perfect-fluid obeys EOS(2) for $k \neq 0$ and $\gamma \neq 0, 1$, we find from Eqs. (35) and (36) that

$$\alpha = \gamma, \quad p_1 = 0, \quad p_2 = \frac{k}{{m_b}^{\gamma}(8\pi G)^{(\gamma-1)}\gamma^2}\xi^{-2\gamma}{\rho_1}^{\gamma} = (\gamma-1)\rho_2, \quad [Case\ III] \tag{50}$$

or

$$\alpha = \frac{1}{\gamma}, \quad p_2 = 0, \quad p_1 = \frac{k}{{m_b}^{\gamma}(8\pi G)^{(\gamma-1)}\gamma^{2\gamma}}\xi^2{\rho_2}^{\gamma} = (\gamma-1)\rho_1 \quad [Case\ IV]. \tag{51}$$

In all the above cases, we obtain contradictions to the basic equations, so we can conclude that there is no solution in the above-mentioned cases.

B. Equation of State (3)

If a perfect-fluid satisfies EOS(3), Eqs. (35) and (36) yield

$$p_1 = k\rho_1, \quad p_2 = k\rho_2 \quad [Case\ V]. \tag{52}$$

When $k = -1$, we have a contradiction in the basic Eqs. (37)-(47). For $k \neq -1$, we assume that $\rho_1 \neq 0$ and $\rho_2 \neq 0$. In this case, with Eqs. (37) and (38), it follows that $-2\alpha\rho_1\rho_2 - \dot{\rho_2}\rho_1 + \dot{\rho_1}\rho_2 = 0$, and from Eqs. (39) and (40), we obtain $-2\rho_1\rho_2 - \dot{\rho_2}\rho_1 + \dot{\rho_1}\rho_2 = 0$. These two expressions imply that $\rho_1\rho_2 = 0$ as $\alpha \neq 1$. For the case when $\rho_1 = 0 = p_1$ and $\rho_2 \neq 0$, we have a contradiction. The case when $\rho_2 = 0 = p_2$ and $\rho_1 \neq 0$, Eq. (43) implies that $\dot{\lambda} = 0$, *i.e.*, $\lambda = constant$. Now Eq. (42) shows that $\rho_1 = -1$, and EOS(3) implies that $p_1 = -k$. Using this information in the basic equations, we get two values of $\dot{\nu}$ from Eq. (39) and Eq. (44). Comparing these values, we get $k = 1$, from which it follows that $\nu = \ln(\frac{b_0}{\xi})$ from both equations. On substituting the values of k and ν in Eq. (46), we reach a contradiction. Hence, there is no self-similar solution in this case.

3. Self-similarity of the Zeroth Kind

This section is devoted to a discussion of the self-similar solutions of the zeroth kind. In this case, the EFEs indicate that the quantities ρ and p must be of the forms

$$\kappa\rho = \frac{1}{x^2}[\rho_1(\xi) + x^2\rho_2(\xi)], \tag{53}$$

$$\kappa p = \frac{1}{x^2}[p_1(\xi) + x^2 p_2(\xi)], \tag{54}$$

where the self-similar variable is $\xi = \frac{x}{e^{-t}}$. If it is assumed that the EFEs and the equations of motion for the matter field are satisfied for the $O[(x)^0]$ and the $O[(x)^2]$ terms separately, we obtain the following set of ordinary differential equations:

$$\dot{\rho_1} = -2\dot{\lambda}(\rho_1 + p_1), \tag{55}$$

$$\dot{\rho_2} = -2\dot{\lambda}(\rho_2 + p_2), \tag{56}$$

$$\dot{p_1} - 2p_1 = \dot{\nu}(\rho_1 + p_1), \tag{57}$$

$$\dot{p_2} = \dot{\nu}(\rho_2 + p_2), \tag{58}$$

$$0 = \ddot{\lambda} + \dot{\lambda}^2 + \dot{\lambda} - \dot{\lambda}\dot{\nu}, \tag{59}$$

$$-\rho_1 = 1 + 4\dot{\lambda} + 3\dot{\lambda}^2 + 2\ddot{\lambda}, \tag{60}$$

$$\rho_2 = \dot{\lambda}^2 e^{-2\nu}, \tag{61}$$

$$p_1 = 1 + 2\dot{\lambda} + \dot{\lambda}^2 + 2\dot{\nu} + 2\dot{\lambda}\dot{\nu}, \tag{62}$$

$$-e^{2\nu} p_2 = 2\ddot{\lambda} + 3\dot{\lambda}^2 - 2\dot{\lambda}\dot{\nu}, \tag{63}$$

$$p_1 = \ddot{\lambda} + \dot{\lambda}^2 + \dot{\lambda} + \dot{\lambda}\dot{\nu} + \ddot{\nu} + \dot{\nu}^2, \tag{64}$$

$$-e^{2\nu} p_2 = \ddot{\lambda} + \dot{\lambda}^2 - \dot{\lambda}\dot{\nu}. \tag{65}$$

A. EOS(1) and EOS(2)

These two EOS for the zeroth kind give contradictions; hence, no solution exists.

B. EOS(3)

When we take a perfect-fluid satisfying EOS(3), it follows from Eqs. (53) and (54) that

$$p_1 = k\rho_1, \quad p_2 = k\rho_2. \tag{66}$$

We can proceed in a similar way as in the case of self-similarity of the second kind with EOS(3) and meet a contradiction in each case.

4. Self-similarity of the Infinite Kind

In this section, we discuss the self-similar solution of the infinite kind. For this case, the EFEs imply that the quantities ρ and p must be of the forms

$$\kappa\rho = \frac{1}{t^2}\rho_1(\xi) + \frac{1}{c^2}\rho_2(\xi), \tag{67}$$

$$\kappa p = \frac{1}{t^2}p_1(\xi) + \frac{1}{c^2}p_2(\xi), \tag{68}$$

where $\xi = \frac{e^{\frac{x}{c}}}{t}$. Now if we require that the EFEs and the equations of motion for the matter field are satisfied for the $O[(t)^0]$ and the $O[(t)^{-2}]$ terms separately, we obtain a set of ordinary differential equations. For a perfect fluid, Eqs. (7)-(12) take the forms

$$\dot{\rho_1} + 2\rho_1 = -2\dot{\lambda}(\rho_1 + p_1), \tag{69}$$

$$\dot{\rho_2} = -2\dot{\lambda}(\rho_2 + p_2), \tag{70}$$

$$\dot{p_1} = \dot{\nu}(\rho_1 + p_1), \tag{71}$$

$$\dot{p_2} = \dot{\nu}(\rho_2 + p_2), \tag{72}$$

$$0 = \ddot{\lambda} + \dot{\lambda}^2 - \dot{\lambda}\dot{\nu}, \tag{73}$$

$$\rho_1 = \dot{\lambda}^2 e^{-2\nu}, \tag{74}$$

$$\rho_2 = -3\dot{\lambda}^2 - 2\ddot{\lambda}, \tag{75}$$

$$-e^{2\nu}p_1 = 2\ddot{\lambda} + 3\dot{\lambda}^2 + 2\dot{\lambda} - 2\dot{\lambda}\dot{\nu}, \tag{76}$$

$$p_2 = \dot{\lambda}^2 + 2\dot{\lambda}\dot{\nu}, \tag{77}$$

$$-e^{2\nu}p_1 = \ddot{\lambda} + \dot{\lambda}^2 + \dot{\lambda} - \dot{\lambda}\dot{\nu}, \tag{78}$$

$$p_2 = \ddot{\lambda} + \dot{\lambda}^2 + \dot{\lambda}\dot{\nu} + \ddot{\nu} + \dot{\nu}^2, \tag{79}$$

respectively.

A. EOS(1) and EOS(2)

When a perfect-fluid satisfies EOS(1), it can be seen from Eq. (67) and Eq. (68) that

$$p_1 = 0 = \rho_1, \quad p_2 = k(8\pi G)^{(1-\gamma)}{\rho_2}^{\gamma} \quad [Case\ I]. \tag{80}$$

For the condition given by EOS(2), it turns out that

$$p_1 = 0 = \rho_1, \quad p_2 = \frac{k}{{m_b}^{\gamma}(8\pi G)^{(\gamma-1)}}(\rho_2 - \frac{p_2}{(\gamma-1)})^{\gamma} \quad [Case\ II]. \tag{81}$$

In both cases, Eq. (74) shows that $\lambda = constant$; then, Eq. (75) gives $\rho_2 = 0$. Now, EOS(1) and EOS(2) show that $p_2 = 0$. We are left with Eq. (79), which gives $\ddot{\nu} + \dot{\nu}^2 = 0$ and, consequently, $\nu = \ln[c_1(\ln\xi - c_2)]$. Finally, we have the following vacuum solution:

$$\nu = \ln[ax - b\ln t - c], \quad \lambda = contant,$$
$$\rho_1 = 0 = p_1, \quad \rho_2 = 0 = p_2. \tag{82}$$

B. EOS(3)

It follows from Eqs. (67) and (68) that this equation of state gives

$$p_1 = k\rho_1, \quad p_2 = k\rho_2 \quad [Case\ III]. \tag{83}$$

We consider the following two possibilities: $k = -1$ and $k \neq -1$. In the first case, we have

$$p_1 + \rho_1 = 0, \quad p_2 + \rho_2 = 0. \tag{84}$$

If we make use of Eqs. (69)-(72), we obtain $\rho_1 = 0 = p_1$ and $\rho_2 = -p_2 = constant$. Then, it follows from Eq. (74) that $\lambda = constant$ and Eq. (75) gives $\rho_2 = 0$. Also, from EOS(3), we can say that $p_2 = 0$. This turns out to be exactly the same solution as in the cases of EOS(1) and EOS(2).

In the second case, *i.e.*, $k \neq -1$ when $\rho_1 \neq 0$ and $\rho_2 \neq 0$, solving Eqs. (69)-(72) simultaneously, we have $\rho_1\rho_2 = 0$. If we consider $\rho_1 = 0 = p_1$ or $\rho_2 = 0 = p_2$, we again have the same results as in EOS(1) and EOS(2).

IV. TILTED DUST CASE

1. Self-similarity of the First Kind

If we set $p_1 = 0 = p_2$ in the basic Eqs. (24)-(34) for the tilted perfect-fluid case with self-similarity of the first kind, Eqs. (30) and (32) immediately give $\lambda = 0$. From the rest of the equations, Eq. (26) and Eq. (27) give rise to two cases: either $\nu = constant$ or $\rho_1 = 0 = \rho_2$. The first possibility contradicts Eq. (31) and the second possibility contradicts Eq. (28). Hence, no solution exists for this case.

2. Self-similarity of the Second Kind

When we take $p_1 = 0 = p_2$ in Eqs. (37)-(47) for the tilted perfect-fluid case with self-similarity of the second kind, Eqs. (39) and (40) immediately give either $\nu = constant$ or $\rho_1 = 0 = \rho_2$. This again leads to a contradiction.

3. Self-similarity of Zeroth Kind

In this case, we also have a contradiction; consequently, there is no solution.

4. Self-similarity of Infinite Kind

In this case we take $p_1 = 0 = p_2$ in Eqs. (69)-(79) for the tilted perfect-fluid case with self-similarity of the infinite kind. Eqs. (71) and (72) imply that either $\nu = constant$ or $\rho_1 = 0 = \rho_2$. In the first case, Eq. (77) directly gives $\lambda = constant$, and Eqs. (74) and (75) show that $\rho_1 = 0 = \rho_2$; hence, we have a Minkowski spacetime. For the second case, Eq. (74) implies that $\lambda = constant$, and we are left with Eq. (79) only, which gives $\ddot{\nu} + \dot{\nu}^2 = 0$. This yields exactly the same result as for a perfect-fluid with self-similarity of the infinite kind using EOS(1) and EOS(2).

V. ORTHOGONAL PERFECT FLUID CASE

1. Self-similarity of the First Kind

Here, we discuss self-similar solutions for the orthogonal perfect-fluid case. Firstly, we consider self-similarity of the first kind. In this case, the self-similar variable is $\xi = t$, and the plane symmetric spacetime becomes

$$ds^2 = x^2 e^{2\nu(t)} dt^2 - dx^2 - x^2 e^{2\lambda(t)}(dy^2 + dz^2). \quad (85)$$

The EFEs and the equations of motion for the matter field give the following set of equations:

$$e^{2\nu}(1 + \rho) = \lambda'^2, \quad (86)$$
$$e^{2\nu}(3 - p) = 3\lambda'^2 + 2\lambda'' - 2\lambda'\nu', \quad (87)$$
$$e^{2\nu}(1 - p) = \lambda'' + \lambda'^2 - \lambda'\nu', \quad (88)$$
$$2\lambda'(\rho + p) = -\rho_1', \quad (89)$$
$$\rho + 3p = 0, \quad (90)$$

where a prime represents a derivative with respect to $\xi = t$. Eq. (90) gives us an equation of state for this system of equations. Using this EOS in Eq. (89), we can get the value of λ' in terms of pressure as $\lambda' = -\frac{3p'}{4p}$. Also, using this value of λ' in Eq. (86), we can have the value of $e^{2\nu}$ in terms of pressure as $e^{2\nu} = \frac{9{p'}_1^2}{16p_1^2(1-3p_1)^2}$. Solving Eqs. (87) and (88) simultaneously, we get a contradiction to the value of $e^{2\nu}$. This implies that no solution exists for this case.

2. Self-similarity of the Second Kind

Now, we consider a self-similarity of the second kind. For this case, the self-similar variable is given by t. The plane symmetric spacetime takes the form

$$ds^2 = x^{2\alpha} e^{2\nu(t)} dt^2 - dx^2 - x^2 e^{2\lambda(t)}(dy^2 + dz^2). \quad (91)$$

The EFEs imply that the quantities ρ and p must be of the forms

$$\kappa\rho = x^{-2}\rho_1(\xi) + x^{-2\alpha}\rho_2(\xi), \quad (92)$$
$$\kappa p = x^{-2}p_1(\xi) + x^{-2\alpha}p_2(\xi), \quad (93)$$

where $\xi = t$. We note that the solution is always singular at $x = 0$, which corresponds to the physical center. When the EFEs and the equations of motion for the matter field are satisfied for the $O[(x)^0]$ and the $O[(x)^{-2-2\alpha}]$ terms separately, we obtain a set of ordinary differential equations. These are given as

$$\rho_1 = -1, \quad (94)$$
$$\rho_2 = e^{-2\nu}\lambda'^2, \quad (95)$$
$$0 = (1 - \alpha)\lambda', \quad (96)$$
$$p_1 = 1 + 2\alpha, \quad (97)$$
$$e^{2\nu}p_2 = -2\lambda'' + 2\lambda'\nu' - 3\lambda'^2, \quad (98)$$
$$p_1 = \alpha^2, \quad (99)$$
$$-e^{2\nu}p_2 = \lambda'' - \lambda'\nu' + \lambda'^2, \quad (100)$$
$$2\lambda'(\rho_1 + p_1) = -\rho_1', \quad (101)$$
$$2\lambda'(\rho_2 + p_2) = -\rho_2', \quad (102)$$
$$\alpha(\rho_1 + p_1) = -2p_1, \quad (103)$$
$$\rho_2 + 3p_2 = 0. \quad (104)$$

Eq. (104) gives us an EOS for this system. Since $p_1 = 0$ contradicts Eq. (99), and $\rho_1 = 0$ contradicts Eq. (94), a vacuum spacetime is not compatible with this case. Eq. (96) gives us $\lambda = constant$, and Eq. (95) shows that $\rho_2 = 0$. From Eq. (104), we have $p_2 = 0$, and Eqs. (97) and (99) yield $\alpha = 1 \pm \sqrt{2}$ and, consequently, $p_1 = 3 \pm 2\sqrt{2}$. These values contradict our basic equations; hence we have no solution in this case, as well.

3. Self-similarity of the Zeroth Kind

The self-similar variable for the zeroth kind is also t, and the metric for the plane symmetry becomes

$$ds^2 = e^{2\nu(t)} dt^2 - dx^2 - x^2 e^{2\lambda(t)}(dy^2 + dz^2). \quad (105)$$

In the case of self-similarity of the zeroth kind, the basic equations for a perfect-fluid gives us a contradiction; hence, we have no solution in this case.

4. Self-similarity of the Infinite Kind

For the self-similarity of the infinite kind, the self-similar variable is $\xi = t$. The metric for this kind takes the form

$$ds^2 = e^{2\nu(t)} dt^2 - dx^2 - e^{2\lambda(t)}(dy^2 + dz^2). \quad (106)$$

A set of ordinary differential equations in terms of ξ is obtained from the EFEs and the equations of motion for the matter field:

$$2\lambda'(\rho + p) = -\rho', \quad (107)$$
$$\rho = \lambda'^2 e^{-2\nu}, \quad (108)$$
$$-e^{2\nu}p = 2\lambda'' + 3\lambda'^2 - 2\lambda'\nu', \quad (109)$$
$$-e^{2\nu}p = \lambda'' + \lambda'^2 - \lambda'\nu'. \quad (110)$$

We use EOS(3) to solve the set of equations, Eqs. (107)-(110), as this is the only compatible equation of state for this kind. This equation of state is given by

$$p = k\rho. \tag{111}$$

Using EOS(3) in Eqs. (109) and (110), we have

$$-e^{2\nu}k\rho = 2\lambda'' + 3\lambda'^2 - 2\lambda'\nu', \tag{112}$$

$$-e^{2\nu}k\rho = \lambda'' + \lambda'^2 - \lambda'\nu'. \tag{113}$$

Now, putting the value of λ'^2 in the above two equations from Eq. (108) and using Eqs. (112) and Eq. (113), we have

$$-e^{2\nu}\rho(k+1) = 0. \tag{114}$$

This gives two possibilities either $\rho = 0$ or $k = -1$. For the first possibility, we immediately get from EOS(3) that $p = 0$ and from Eq. (108) $\lambda = constant$. Thus, we obtain the following vacuum solution:

$$\nu = \nu(t), \quad \lambda = a_0, \quad \rho = 0 = p. \tag{115}$$

For the second possibility, EOS(3) becomes $\rho + p = 0$. Solving Eqs. (107)-(110) simultaneously, we obtain the same solution as we did for the first possibility.

VI. ORTHOGONAL DUST CASE

1. Self-similarity of the First Kind

In this case, we substitute $p = 0$ in the basic equations for the orthogonal perfect-fluid case with a self-similarity of first kind. Eq. (90) immediately shows that the resulting spacetime must be vacuum. Eq. (86) gives $e^{2\nu} = \lambda'^2$. Also, on solving Eqs. (87) and (88), we reach $e^{2\nu} = \lambda'^2$. For this case, we have the solution

$$\lambda = \int e^{\nu}dt, \quad \rho = 0 = p. \tag{116}$$

For $\nu = 0$, this leads to the solution

$$\lambda = t, \quad \nu = 0, \quad \rho = 0 = p. \tag{117}$$

2. Self-similarity of the Second and Zeroth Kinds

For self-similarity of the second and the zeroth kinds we arrive at contradictions; hence, no solutions exist.

3. Self-similarity of the Infinite Kind

We set $p = 0$ in the basic equations for the orthogonal perfect-fluid case for a self-similarity of an infinite kind. Eqs. (109) and (110) give $\lambda = constant$. Now, Eq. (108) ensures us that the resulting spacetime must be vacuum. Hence, we have the same solution as in the case of an orthogonal perfect-fluid case with a self similarity of an infinite kind. This is given by

$$\nu = \nu(t), \quad \lambda = a_0, \quad \rho = 0 = p. \tag{118}$$

VII. PARALLEL PERFECT-FLUID CASE

Since we do not have self-similar variables for the first, second, and zeroth kinds of KSS in the parallel perfect-fluid case, hence no solution exists for these kinds of KSS.

For a self-similarity of the infinite kind, the self-similar variable is $\xi = x$. The metric for the plane symmetry of the infinite kind reduces to

$$ds^2 = e^{2\nu(x)}dt^2 - dx^2 - e^{2\lambda(x)}(dy^2 + dz^2). \tag{119}$$

A set of ordinary differential equations in terms of ξ is obtained from the EFEs and the equations of motion for the matter field:

$$-\rho = 3\lambda'^2 + 2\lambda'', \tag{120}$$

$$p = \lambda'^2 + 2\lambda'\nu', \tag{121}$$

$$p = \lambda'' + \lambda'^2 + \lambda'\nu' + \nu'' + \nu'^2, \tag{122}$$

$$p' = \nu'(\rho + p). \tag{123}$$

Here, a prime ($'$) represents a derivative with respect to $\xi = x$. We use EOS(3) to solve the set of equations, Eqs. (120)-(123). The equation of state is given by

$$p = k\rho. \tag{124}$$

Clearly Eq. (123) gives the value of ν' in terms of ρ as $\nu' = \frac{k\rho'}{(k+1)\rho}$. Now, we assume, for the sake of simplicity, that ρ is a linear function of ξ. When we use this assumption in Eq. (120), we obtain a value of λ, but Eq. (121) gives a contradiction. Hence, no self-similar solution exists for the parallel perfect-fluid case.

VIII. PARALLEL DUST CASE

Again, we do not have any self-similar variables for the first, second, and zeroth kinds of KSS in this case. Consequently, no solutions exist for these kinds of KSS.

When we set $p = 0$ in the basic equations for the parallel perfect-fluid case with a self-similarity of the infinite kind, Eq. (123) immediately shows that either $\nu = constant$ or $\rho = 0$. In the first case, Eqs. (120) and (121) show that the resulting spacetime is *Minkowski*. In

Table 1. Perfect-fluid kinematic self-similar solutions.

Self-similarity	**Solution**
First kind (tilted)	None
First kind (orthogonal)	None
First kind (parallel)	None
Second kind (tilted)(EOS(1))	None
Second kind (tilted)(EOS(2))	None
Second kind (tilted)(EOS(3))	None
Second kind (orthogonal)	None
Second kind (parallel)	None
Zeroth kind (tilted)(EOS(1))	None
Zeroth kind (tilted)(EOS(2))	None
Zeroth kind (tilted)(EOS(3))	None
Zeroth kind (orthogonal)	None
Zeroth kind (parallel)	None
Infinite kind (tilted)(EOS(1))	vacuum solution given by Eq. (82)
Infinite kind (tilted)(EOS(2))	vacuum solution given by Eq. (82)
Infinite kind (tilted)(EOS(3))	vacuum solution given by Eq. (82)
Infinite kind (orthogonal)(EOS(3))	vacuum solution given by Eq. (115)
Infinite kind (parallel)(EOS(3))	None

the second case, where $\rho = 0$, Eq. (121) shows that either $\lambda' = 0$ or $\lambda' = -2\nu'$. When $\lambda' = 0$, Eq. (122) gives the following vacuum solution:

$$\nu = \ln(c_1(\xi - c_2)), \quad \lambda = constant = c_0, \\ \rho = 0 = p. \tag{125}$$

For the case $\lambda' = -2\nu'$, Eqs. (120) and (122) gives the following vacuum solution:

$$\nu = -\frac{1}{3}\ln(3\xi - 2c_1), \quad \lambda = \ln(c_2(3\xi - 2c_1)^{\frac{2}{3}}), \\ \rho = 0 = p. \tag{126}$$

IX. CONCLUSION

We have classified KSS perfect-fluid and dust solutions for the cases when the KSS vector is tilted, orthogonal, and parallel to the fluid flow by using EOS(1), EOS(2) and EOS(3). In most cases, we solve the governing equations to get solutions, but there are a few exceptions. We obtain a total of six independent plane symmetric self-similar solutions. The parallel case gives contradictions for the first, second, and zeroth kinds of KSS hence, no self-similar solutions exist in these cases.

For the tilted perfect-fluid case with a self-similarity of the first kind, we have contradictory results; hence, no solutions exist in this case. For self-similarity of the second and zeroth kinds with EOS(1) and EOS(2), we again reach contradictions. For KSS of the second and the zeroth kinds with EOS(3), we have contradictions in all cases. For self-similarity of the infinite kind, we find that the spacetime must be vacuum for EOS(1) and EOS(2). A vacuum solution also turns up for EOS(3) when $k = -1$. For the case when $k \neq -1$ and either $\rho_1 = 0$ or $\rho_2 = 0$, we again have a vacuum spacetime.

In the tilted dust case with a self-similarity of the infinite kind, we obtain two possibilities. One possibility gives a vacuum solution, and the other yields a Minkowski spacetime. There is no solution for any other kind of self-similarity.

For the orthogonal perfect-fluid case with a self-similarity of the first kind, we have no solution. In the orthogonal perfect-fluid case with KSS of the second kind and the zeroth kind, we have contradictions; hence, no solutions exist for these cases. For the infinite kind of self-similarity with EOS(3), we have a vacuum solution with arbitrary ν and constant λ.

In the orthogonal dust case with a self-similarity of the first kind, we have a vacuum spacetime, where ν and λ are related to each other. For $\nu = 0$, λ simply becomes t. We have contradictory results for KSS of the second and the zeroth kinds; hence, no solutions exist in this case. We obtain the same solution for the infinite kind of self-similarity as for the orthogonal perfect-fluid with KSS of an infinite kind. In the parallel perfect-fluid case, no self-similar solution exists. However, we obtain three different solutions for the parallel dust case with a self-similarity of the infinite kind.

We would like to mention here that this paper has focused on a classification of plane symmetric kinematic self-similar solutions under certain restrictions. A classification for the most general plane symmetric kinematic

Table 2. Dust kinematic self-similar solutions.

Self-similarity	**Solution**
First kind (tilted)	None
First kind (orthogonal)	solution given by Eq. (116)
First kind (parallel)	None
Second kind (tilted)	None
Second kind (orthogonal)	None
Second kind (parallel)	None
Zeroth kind (titled)	None
Zeroth kind (orthogonal)	None
Zeroth kind (parallel)	None
Infinite kind (tilted) (case 1)	Minkowski
Infinite kind (tilted) (case 2)	vacuum solution given by Eq. (82)
Infinite kind (orthogonal)	vacuum solution given by Eq. (115)
Infinite kind (parallel)(case 1)	Minkowski
Infinite kind (parallel)(case 2)	vacuum solution given by Eq. (125)
Infinite kind (parallel)(case 3)	vacuum solution given by Eq. (126)

self-similar solutions is under investigation [23] and will appear elsewhere. The results can be summarized in the form of the tables given above.

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