Journal of the Korean Physical Society, Vol. 49, No. 1, July 2006, pp. 15∼20

Statistical Entropy and Superradiance in (2+1)-Dimensional Acoustic Black Holes

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(Received 27 February 2006)

We study the "draining bathtub" as an acoustic analogue of a three-dimensional rotating black hole. The rotating fluid near the sonic horizon necessarily gives rise to superradiant modes, which are partially responsible for the thermodynamic quantities in this rotating vortex-like hole. Using the improved brick-wall method, we explicitly calculate the free energy of the system by treating the superradiance carefully and obtain the desired entropy formula.

PACS numbers: 04.70.Dy, 04.80.-y
Keywords: Superradiance, Entropy, Black hole

I. INTRODUCTION

As suggested by Bekenstein, a black hole may have an intrinsic entropy proportional to the surface area at the event horizon [1]. Subsequently Hawking provided quantum field theoretic calculations for the Schwarzschild black hole [2]. Since then, much attention has been paid to the statistical-mechanical origin of the entropy, especially for rotating black holes [3–7]. In the brick-wall method, quantum effects can be easily taken into account [8]. Introducing a brick-wall cutoff makes it possible to remove the divergent term due to the infinite blue shift near the horizon [9–11]. The entropy from the brick-wall method consists of mainly two parts: the most dominant term compared to the logarithmic one gives the Bekenstein-Hawking entropy, and the other represents a typical infrared contribution at large distances. Although this original brick-wall method is useful for various models [8–18], some difficulties may arise because a thermal equilibrium between the black hole and the external field, even in a large spatial region, is assumed to exist. Obviously, this method cannot be applied to a nonequilibrium system, such as a system of non-stationary space-time with two horizons, because the two horizons have different temperatures and the thermodynamical laws are also invalid there. A thin-film method, an improved brick-wall method, has been introduced to solve these problems [19, 20]. In the thin layer, local thermal equilibrium exists, and the divergent term due to large distance does no longer appears.

On the other hand, in Ref. [21], many black-hole issues have been treated as field theoretical problems in fluid because this acoustic analogue is useful in that its thermodynamics, such as the Hawking radiation and entropy, might be tested hopefully in the laboratory. Moreover, a "draining bathtub", referred to as an acoustic analogue of a rotating black hole, has been well defined [22–27]. However, the conventional brick-wall cannot be applied in this model because it is impossible for the angular velocities of particles to have the same constant value in the whole region while for the Bañados-Teitelboim-Zanelli (BTZ) black hole [28], the method can be used effectively to examine some results from superradiance [16]. Thus, in this paper, we would like to investigate the draining bathtub in terms of the thin-film method, which is helpful because the angular speed near the horizon can be approximately fixed to a constant. In Sec. II, the generic formulation of the free energy for a rotating black hole is given in the grand canonical ensemble, and it will be shown precisely why the thin-film method should be used in our model. Then, in Sec. III, the thermodynamic quantities are calculated by treating superradiant and nonsuperradiant(regular) modes more carefully. Finally, a summary and discussion are given in Sec. IV.

II. FORMULATION OF THE FREE ENERGY FOR A ROTATING BLACK HOLE

In order to calculate the entropy of a given system in the original brick-wall method, we consider a quantum

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gas of scalar particles confined within a box near the horizon of a black hole and introduce a cut-off parameter as in Ref. [8]. The free scalar field is assumed to satisfy the Klein-Gordon equation given by $(\Box + m^2)\psi = 0$ with boundary conditions

$$\psi(r_H + h) = \psi(L) = 0, \tag{1}$$

where r_H is the horizon, and $r_H + h$ and L represent the inner and the outer walls of a "spherical" box, respectively, and h is an infinitesimal cutoff parameter. Suppose that this system is in thermal equilibrium at a temperature $T = \beta^{-1}$ with an external reservoir. Then, in a stationary rotating axisymmetric black hole, a partition Z for an ideal bosonic gas in the grand canonical ensemble is given by [16,29]

$$\ln Z = \sum_\lambda \ln \sum_{k=0}^{\infty} [e^{-\beta(\epsilon_\lambda - \Omega j_\lambda)}]^k, \tag{2}$$

where $k = 0, 1, 2, \cdots$ is an occupation number, ϵ_λ and j_λ denote the energy and the angular momentum eigenvalues for a single-particle state λ, respectively, and Ω is the angular speed in equilibrium. The series in the partition function in Eq. (2) has a finite value for $\epsilon_\lambda - \Omega j_\lambda > 0$, but it becomes divergent for $\epsilon_\lambda - \Omega j_\lambda < 0$, so it is ill-defined. In order to resolve such a problem caused by the rotation of the geometry, we deal with the mode solutions of the Klein-Gordon field carefully, which will be of the form $\psi(t, r, \phi) \sim e^{-i\omega t + im\phi}$ for an observer at rest at infinity (ROI) because there exist two Killing vector fields denoted by ∂_t and ∂_ϕ.

Note that the angular speed Ω in Eq. (2) appears in the thermodynamic first law for a reservoir; *i.e.*, $dE = TdS + \Omega dJ$ for a stationary rotating system. Besides, the angular speed of a particle for a ROI should be restricted because no particles can move faster than the speed of light. In fact, it takes a value between the maximum Ω_+ and the minimum Ω_- given by

$$\Omega_\pm(r) = \Omega_0(r) \pm \sqrt{(\partial_t \cdot \partial_\phi / \partial_\phi \cdot \partial_\phi)^2 - \partial_t \cdot \partial_t / \partial_\phi \cdot \partial_\phi}, \tag{3}$$

where $\Omega_0(r)$ is the angular speed of a zero-angular-momentum-observer(ZAMO) [16]. It is clear that both $\Omega_\pm$ converge to a constant value of $\Omega_H \equiv \Omega_0(r = r_H)$ near the horizon, so the angular speed of every particle near the horizon can be always thought of as Ω_H. Since the dominant contribution to the physical quantities of the system, such as the total entropy, is attributed to the quantum gas in the vicinity of the horizon, it is natural to assume that the system is in equilibrium with a uniform angular speed $\Omega = \Omega_H$.

Before formulating a generic free energy for a rotating black hole, we introduce the density function defined by $g(\omega, m) = \partial n(\omega, m)/\partial\omega$ in order to calculate the free energy strictly, where $n(\omega, m)$ is the number of mode solutions whose frequencies, or energies, are all below ω for a given value of angular momentum m. Thus, $g(\omega, m)d\omega$ represents the number of single-particle states whose energies lie between ω and $\omega + d\omega$ and whose angular momenta are m. Now, from the partition function in Eq. (2), the free energy F is obtained as

$$\beta F = -\ln Z = -\sum_m \int d\omega g(\omega, m) \times \ln \sum_k \left[e^{-\beta(\omega - \Omega_H m)}\right]^k. \tag{4}$$

It would be plausible to mention here that a ZAMO near the horizon ($r \approx r_H$) could measure only ingoing modes given by $\psi_{\text{in}}(x) \sim e^{-i\tilde{\omega}\tilde{t} + i\tilde{m}\tilde{\phi}}$ while a ROI would see both ingoing and outgoing ones, where $\tilde{t} = t$, $\tilde{\phi} = \phi - \Omega_H t$, $\tilde{\omega} = |\omega - \Omega_H m| > 0$, and $\tilde{m} = \text{sgn}(\omega - \Omega_H m)m$. Here, $\text{sgn}(x)$ is 1 for $x > 0$ and -1 for $x < 0$. The ingoing wave near the horizon consists of two parts: one is the so-called superradiant (SR) modes with $\omega - \Omega_H m < 0$, and the other is the nonsuperradiant (NS) modes with $\omega - \Omega_H m > 0$. Then, $e^{-i\tilde{\omega}t + i\tilde{m}\tilde{\phi}} = e^{i\omega t - im\phi}$ for SR modes, and $e^{-i\tilde{\omega}t + i\tilde{m}\tilde{\phi}} = e^{-i\omega t + im\phi}$ for NS modes. Since only the ingoing modes are considered near the horizon, (ϵ, j) has the value of (ω, m) for single-particle states with NS modes while (ϵ, j) becomes $(-\omega, -m)$ for SR ones. Separating the SR modes from the NS ones, the free energy in Eq. (4) should be replaced by $F = F_{\text{NS}} + F_{\text{SR}}$ with

$$\beta F_{\text{NS}} = \sum_m \int_{\omega > \Omega_H m} d\omega g(\omega, m) \ln\left[1 - e^{-\beta(\omega - \Omega_H m)}\right], \tag{5}$$

$$\beta F_{\text{SR}} = \sum_m \int_{\omega < \Omega_H m} d\omega g(\omega, m) \ln\left[1 - e^{\beta(\omega - \Omega_H m)}\right]. \tag{6}$$

Note that ω is positive definite and that the density functions are given by $g(\omega, m) = \partial n(\omega, m)/\partial\omega$ for the NS modes and $g(\omega, m) = -\partial n(\omega, m)/\partial\omega$ for the SR ones. Both Eqs. (5) and (6) can be obtained from $\beta F = -\sum_{\tilde{m}} \int d\tilde{\omega} g(\tilde{\omega}, \tilde{m}) \ln \sum_k \exp(-k\beta\tilde{\omega})$, where $g(\tilde{\omega}, \tilde{m}) = \partial n/\partial\tilde{\omega}$.

On the other hand, the angular speed of any particle cannot reach Ω_H above a critical radius in our model because of the restriction on Ω from Eq. (3), which will be explicitly shown in the following section, so that global thermal equilibrium cannot be achieved. Instead, if we consider scalar particles confined within a thin layer near the horizon [19,20], their angular speeds naturally take the same value of a constant Ω_H due to local thermal equilibrium, and the thin-layer method is useful for finding the thermodynamic quantities in our model.

Apparently, the degrees of freedom of a field within a thin layer near the horizon play a major role in the calculation of the entropy of a black hole; hence, global thermal equilibrium is not necessary anymore because particles are assumed to be distributed only in a narrow region. Since it is well known that the Hawking radiation is derived from the vacuum fluctuation near the horizon, the Bekenstein-Hawking entropy should be associated with the field in this narrow region, where thermal equilibrium exists locally and statistical mechanics remains valid. This local thermal equilibrium is the main postulate of the thin-film method, and the thermodynamic properties, such as the pressure and the temperature, near the horizon are assumed to vary slightly. The thickness of the layer is assumed to be so small on a macroscopic scale that the physical quantities can be considered to be constant and that the narrow region can be locally in thermal equilibrium. Also, it is assumed to be very large on a microscopic scale to make sure that statistical mechanics remains valid. Specifically, in our model, the outer boundary L of the spherical box in Eq. (1) is replaced by $r_H+h+\delta$, where the parameter δ is a positive, physical, small quantity related to the thickness of the layer. δ has a scale over the Plank length, but the brick-wall cutoff h is very small compared to the Plank length.

III. THERMODYNAMIC QUANTITIES

We now set up an acoustic analogue of a rotating BTZ black hole in order to investigate its thermodynamics with superradiance taken into account. In the irrotational fluid, the propagation of sound waves is governed by an equation of motion [21],

$$\Box\psi = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\psi) = 0, \tag{7}$$

where ψ is the fluctuation of the velocity potential interpreted as a sonic wave function. The metric is given by

$$g_{\mu\nu} = \frac{\rho_0}{c}\begin{pmatrix} -(c^2-v_0^2) & -v_0^i \\ -v_0^j & \delta_{ij}\end{pmatrix} \quad \text{with } i,j = 1,2,3, \tag{8}$$

where c is the speed of sound wave, ρ_0 and v_0^i are the mass density and the velocity of the mean flow, respectively, δ_{ij} is the Kronecker delta, and $v_0^2 = \delta_{ij}v_0^iv_0^j$. Note that the velocity potential is linearized as $\Psi = \psi_0 + \psi$ and $\vec{v}_0 = \vec{\nabla}\psi_0$.

We then consider a draining bathtub fluid flow described as a $(2+1)$-dimensional flow with a sink at the origin. If the metric is stationary and axisymmetric, according to the equation of continuity, Stokes' theorem, and conservation of angular momentum, ρ_0 is constant and $\psi_0(r,\phi) = -A\ln(r/a) + B\phi$, where a, A, and B are arbitrary real positive constants [22]. Then, the velocity of the mean flow is given by $\vec{v}_0 = -\hat{r}(A/r) + \hat{\phi}(B/r)$.

Now, let us consider the draining vortex case with $A \neq 0$. Dropping the position-independent prefactor from the metric in Eq. (8), the acoustic line element for the draining bathtub is obtained as

$$ds^2 = -c^2dt^2 + \left(dr + \frac{A}{r}dt\right)^2 + \left(rd\phi - \frac{B}{r}dt\right)^2, \tag{9}$$

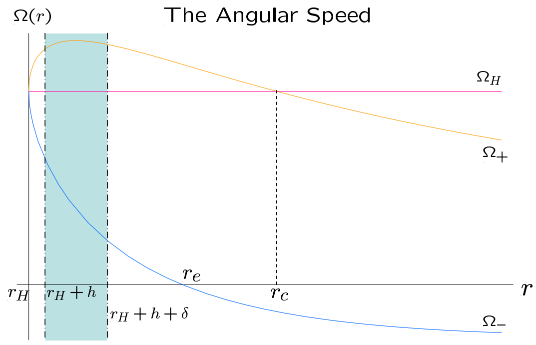

Fig. 1. The possible angular speed of a particle lies between the upper and the lower curves, which denote the maximum and the minimum angular speeds with respect to r, respectively. The shaded area represents the thin layer, which is located inside the ergoregion.

where the radii of the horizon and the ergosphere are

$$r_H = \frac{A}{c}, \qquad r_e = \frac{\sqrt{A^2+B^2}}{c}, \tag{10}$$

respectively. However, the metric in Eq. (9) makes it difficult to calculate thermodynamic quantities because of its (t,r)-component. Fortunately, this can be overcome by a coordinate transformation in the exterior region of $A/c < r < \infty$. Using the transformation given by [26,27]

$$\begin{aligned} dt &\to dt + \frac{Ar}{r^2c^2 - A^2}dr, \\ d\phi &\to d\phi + \frac{AB}{r(r^2c^2-A^2)}dr, \end{aligned} \tag{11}$$

the metric (9) can be rewritten in conventional form as

$$ds^2 = -N^2dt^2 + N^{-2}dr^2 + r^2(d\phi - \Omega_0 dt)^2 \tag{12}$$

with

$$\begin{aligned} N^2(r) &= 1 - \frac{A^2}{c^2r^2} = \frac{r^2 - r_H^2}{r^2}, \\ \Omega_0(r) &= \frac{B}{cr^2} = \Omega_H\frac{r_H^2}{r^2}, \end{aligned} \tag{13}$$

where we rescaled the time coordinate by c for simplicity, and $\Omega_H = B/(cr_H^2)$. Note that the metric in Eq. (12) is similar to that of a rotating BTZ black hole, but the two metrics have a slight difference: although setting $J = 2B/c$ gives the same angular speed $\Omega_0(r)$ as that of the ZAMO, the lapse function $N(r)$ has a different form from that of a BTZ black hole, which is explicitly given by $N^2 = (r^2 - r_+^2)(r^2 - r_-^2)/(r^2l^2)$.

Now, the maximum and the minimum angular speeds are obtained from Eqs. (3) and (12) as

$$\Omega_\pm(r) = \Omega_0(r) \pm \frac{N(r)}{r}. \tag{14}$$

As mentioned before, the angular speed of every particle is Ω_H in the vicinity of the horizon because $\Omega_\pm \to \Omega_H$

as $r \to r_H$. Note that there exists a critical radius $r_c = \sqrt{r_H^2 + \Omega_H^{-2}}$ at which the maximum angular speed Ω_+ is equal to Ω_H. This means that no particle can move along with an angular speed of Ω_H for $r > r_c$, as shown in Fig. 1, so the spherical box should be located inside the critical radius r_c. Moreover, as we discussed in the previous section, the radius of the outer boundary should be smaller than that of the ergosphere. Then, the radial part of the sonic wave satisfies

$$rN^2 \frac{d}{dr}\left[rN^2 \frac{d}{dr}\psi_{\omega m}(r)\right] + r^2 N^4 k^2(r;\omega,m)\psi_{\omega m}(r) = 0, \tag{15}$$

where $k^2(r;\omega,m) = N^{-4}(\omega - \Omega_+ m)(\omega - \Omega_- m)$. It can be easily shown that the function $k(r;\omega,m)$ plays the role of the momentum eigenvalue in the WKB approximation. Therefore, in the thin layer with the range $r_H + h < r < r_H + h + \delta$, the discrete energy ω is related to the total number $n(\omega,m)$ by

$$\pi n(\omega,m) = \int_{r_H+h}^{r_H+h+\delta} dr\, k(r;\omega,m), \tag{16}$$

where $k(r;\omega,m)$ is set to zero if $k^2(r;\omega,m)$ becomes negative for a given (ω,m) [8]. The contribution of k to the following calculations is dominant near the horizon because k is approximately N^{-2} and diverges as r goes to r_H. This tells us that the thin-film method is valid.

We now evaluate the free energy of the total system. The free energy for NS modes, Eq. (5), is written as

$$\begin{aligned}\beta F_{\rm NS} &= \sum_m \int_{\omega > \Omega_H m} d\omega \frac{\partial}{\partial \omega}\left[\frac{1}{\pi}\int_{r_H+h}^{r_H+h+\delta} dr\, k(r;\omega,m)\right] \ln\left[1 - e^{-\beta(\omega - \Omega_H m)}\right] \\ &= -\frac{\beta}{\pi}\int_{r_H+h}^{r_H+h+\delta} dr \sum_m \int d\omega \frac{k(r;\omega,m)}{e^{\beta(\omega-\Omega_H m)} - 1} + \frac{1}{\pi}\int_{r_H+h}^{r_H+h+\delta} dr \sum_m k(r;\omega,m) \ln\left[1 - e^{-\beta(\omega-\Omega_H m)}\right]\Big|_{\omega_{\min}(m)}^{\omega_{\max}(m)}\end{aligned} \tag{17}$$

by using integration by parts with respect to ω. For the sake of convenience, the free energy $F_{\rm NS}$ for NS modes is divided into two parts, which describe states with positive and negative angular momentum, *i.e.*, $F_{\rm NS} = F_{\rm NS}^{(m>0)} + F_{\rm NS}^{(m<0)}$, where

$$F_{\rm NS}^{(m>0)} = -\frac{1}{\pi}\int_{r_H+h}^{r_H+h+\delta} dr N^{-2} \times \int_0^\infty dm \int_{\Omega_+ m}^\infty d\omega \frac{\sqrt{(\omega - \Omega_+ m)(\omega - \Omega_- m)}}{e^{\beta(\omega - \Omega_H m)} - 1}, \tag{18}$$

$$\begin{aligned}F_{\rm NS}^{(m<0)} &= -\frac{1}{\pi}\int_{r_H+h}^{r_H+h+\delta} dr N^{-2} \times \int_{-\infty}^0 dm \int_0^\infty d\omega \frac{\sqrt{(\omega - \Omega_+ m)(\omega - \Omega_- m)}}{e^{\beta(\omega - \Omega_H m)} - 1} \\ &\quad - \frac{1}{\pi\beta}\int_{r_H+h}^{r_H+h+\delta} dr N^{-2} \times \int_{-\infty}^0 dm \sqrt{\Omega_+\Omega_- m^2}\ln(1 - e^{\beta\Omega_H m}).\end{aligned} \tag{19}$$

Similarly, the free energy for SR modes, Eq. (6), becomes

$$\begin{aligned}F_{\rm SR} &= -\frac{1}{\pi}\int_{r_H+h}^{r_H+h+\delta} dr N^{-2} \times \int_0^\infty dm \int_0^{\Omega_- m} d\omega \frac{\sqrt{(\omega - \Omega_+ m)(\omega - \Omega_- m)}}{e^{-\beta(\omega - \Omega_H m)} - 1} \\ &\quad + \frac{1}{\pi\beta}\int_{r_H+h}^{r_H+h+\delta} dr N^{-2} \times \int_0^\infty dm \sqrt{\Omega_+\Omega_- m^2}\ln(1 - e^{-\beta\Omega_H m}).\end{aligned} \tag{20}$$

Fortunately, the second terms of Eqs. (19) and (20) are exactly cancelled, but a large number of tedious calculations is still required. After evaluating the above integrations with respect to ω and m, we obtain the expression of total free energy as

$$F = -\frac{\zeta(3)}{4\beta^3}\int_{r_H+h}^{r_H+h+\delta} dr \frac{(\Omega_+ - \Omega_-)^2}{N^2(\Omega_+ - \Omega_H)^{\frac{3}{2}}(\Omega_H - \Omega_-)^{\frac{3}{2}}}. \tag{21}$$

Then, if Eqs. (13) and (14) are substituted into Eq. (21), the total free energy of our system is reduced to

$$F = -\frac{\zeta(3) r_H^2}{\beta^3}\left[\sqrt{\frac{r_H}{2h}} - \sqrt{\frac{r_H}{2(h+\delta)}} + O(\sqrt{h}, \sqrt{h+\delta})\right] \tag{22}$$

in the leading order. Note that there are no logarithmically divergent terms in the total free energy in Eq. (22) because those are remarkably cancelled as in the rotating BTZ black hole case [16]. In addition, there exists no infrared divergence because the large distance is out of consideration while infrared divergent terms remaining in total free energy were cut off from consideration in Refs. [8] and [16]. It seems appropriate to comment here that for the limiting case of a non-rotating acoustic black hole, $B = 0$, there are no SR modes due to the fact that the angular speed of horizon vanishes, $\Omega_H = 0$; in addition, there is no critical radius r_c and no ergoregion due to $r_e = r_H$. Thus, one might think in this case having only NS modes, that the above result should be recast and give different value; however, the very same result is obtained so that Eq. (22) remains valid in the limit of $\Omega_H \to 0$. Since SR modes are distinguished from NS ones for the ROI and not for the ZAMO near the horizon, it is reasonable for the result from considering the superradiant modes in the rotating acoustic black hole to be the same as the non-rotating one.

On the other hand, the surface gravity is given by $\kappa_H^2 \equiv -\frac{1}{2}\nabla^\mu\chi^\nu\nabla_\mu\chi_\nu\big|_{r=r_H} = 1/r_H^2$, where we used an appropriate Killing field near the horizon, $\chi^\mu = (\partial_t + \Omega_H\partial_\phi)^\mu$ [30]; then, the temperature becomes

$$T_H = \beta_H^{-1} = \frac{\kappa_H}{2\pi} = \frac{1}{2\pi r_H}. \tag{23}$$

Using the thermodynamic relation $S = \beta^2\partial F/\partial\beta|_{\beta=\beta_H} = -3\beta F|_{\beta=\beta_H}$, we obtain the entropy of this system from the free energy in Eq. (22) as

$$S = \frac{3\zeta(3)}{4\pi^2}\left[\sqrt{\frac{r_H}{2h}} - \sqrt{\frac{r_H}{2(h+\delta)}} + O(\sqrt{h}, \sqrt{h+\delta})\right] \tag{24}$$

and it can be rewritten in terms of the thin-wall cutoffs as

$$S = \frac{3\zeta(3)}{8\pi^3}\frac{\bar{\delta}}{\bar{h}(\bar{h}+\bar{\delta})}\ell + O(\bar{h}, \bar{h}+\bar{\delta}), \tag{25}$$

where the cutoffs were defined as $\bar{h} \equiv \int_{r_H}^{r_H+h}\sqrt{g_{rr}}\,dr \approx \sqrt{2r_H h}$ and $\bar{\delta} \equiv \int_{r_H+h}^{r_H+h+\delta}\sqrt{g_{rr}}\,dr \approx \sqrt{2r_H(h+\delta)} - \sqrt{2r_H h}$ in the leading order, and $\ell \equiv \int_0^{2\pi}\sqrt{g_{\phi\phi}}\,d\phi\Big|_{r=r_H} = 2\pi r_H$ is the circumference of the horizon. Note that $\bar{h}$ is called the brick-wall cutoff, and $\bar{\delta}$ represents the thickness of the thin layer. Then, setting $\bar{\delta}/[\bar{h}(\bar{h}+\bar{\delta})] = 16\pi^3/[3\zeta(3)\ell_p]$, or equivalently $\bar{h} = (\bar{\delta}/2)[-1+\sqrt{1+3\zeta(3)\ell_p/(4\pi^3\bar{\delta})}]$, the entropy of sonic wave becomes finite and is equivalent to the Bekenstein-Hawking entropy in the leading order:

$$S = \frac{4\pi r_H}{\ell_p} = S_{\rm BH}, \tag{26}$$

where the three-dimensional Plank length is chosen as $\ell_p \equiv \hbar G/c^3$. It is plausible to make sure that the brick-wall cutoff becomes a universal value of $\bar{h} = [3\zeta(3)/(16\pi^3)]\ell_p$ in the leading order if $\bar{\delta}$ is larger than ℓ_p, which is set to be one in the following calculations.

Next, let us calculate the other thermodynamic quantities, such as the angular momentum of the matter particularly interpreted as a phonon in our model:

$$J_{\rm matter} = -\left.\frac{\partial F}{\partial\Omega_H}\right|_{\beta=\beta_H} = \frac{3\zeta(3)r_H^2\Omega_H}{8\pi^3}\frac{\bar{\delta}}{\bar{h}(\bar{h}+\bar{\delta})}, \tag{27}$$

where the partial derivative was evaluated from Eq. (21). Note that substituting the expression for the cutoff $\bar{h}$ into Eq. (27) gives an angular momentum

$$J_{\rm matter} = 2\Omega_H r_H^2 = \frac{2B}{c}. \tag{28}$$

The internal energy of the system in the frame of a ROI is explicitly calculated as

$$\begin{aligned} E &= F_H + \beta_H^{-1}S + \Omega_H J_{\rm matter} \\ &= \frac{4}{3} + 2r_H^2\Omega_H^2 = \frac{4}{3} + \frac{2B^2}{A^2}, \end{aligned} \tag{29}$$

where $F_H = F|_{\beta=\beta_H}$. In the limit of non-rotating acoustic black holes where $J_{\rm matter} \to 0$, or equivalently $B \to 0$, it can be easily seen that the total energy E has a minimum value of 4/3.

Finally, it seems to be appropriate to comment on the perfect vortex case. If we take the limit of pure spinning acoustic black holes of $A \to 0$, the internal energy in Eq. (29) is undefined. Therefore, we must independently analyze this case whose spacetime represents a fluid with a non-radial flow, but not event horizon exists in this case. Therefore, it is meaningless to consider an analogy between gravitational and acoustic black holes.

IV. DISCUSSION

In this paper, we have studied the thermodynamics of a rotating acoustic black hole involving superradiant modes for the draining vortex case ($A \neq 0$) as an acoustic analogue of a black hole in three dimensions. In order to overcome some difficulties in applying the original brick-wall method to our model, the thin-film method was introduced as an improved brick-wall one. Using this method we obtained thermodynamic quantities such as the free energy, entropy, angular momentum, and internal energy of a thin layer under thermal equilibrium with the black hole. The definition of the thermodynamic black-hole entropy was chosen in Eq. (26) as $S_{BH} = 2\ell/\ell_p$, following that of the BTZ black hole [28], to fix the brick-wall cutoff $\bar{h}$, where the leading order of the entropy becomes $S \approx S_{BH}(1 - \bar{h}/\bar{\delta})$ for a universal value of the brick-wall cutoff. Recovering the physical dimension, the angular momentum in Eq. (28) and the internal energy in Eq. (29) become $J_{\rm matter} = 2c^2B/G$ and $E/c^2 = (4/3 + 2B^2/A^2)c^2/G$, respectively.

In the limit $A \to 0$, the internal energy in Eq. (29) diverges. In fact, the metric in Eq. (12) for $A = 0$ seems to describe a pure rotation without horizons. However, this limit cannot be taken because it has a naked singularity at $r = 0$. Therefore, we should consider a form different from the metric in Eq. (12) in order to deal with the pure rotation. Also, in the pure rotation, the brick-wall method cannot be used to calculate thermodynamic quantities because the particles are distributed over the whole region, and it is impossible for the particles to fix the angular velocities to a special value.

Finally, for the purpose of checking the stability of the system, the heat capacity can be calculated as [31]

$$C_J \equiv \left(\frac{\partial E}{\partial T}\right)_J \bigg|_{\beta=\beta_H} = 2S, \tag{30}$$

where we used the first law of thermodynamics, $dE = TdS + \Omega_H dJ$, and the thermodynamic relation between the entropy and the free energy from Eq. (21). Since the entropy is always positive, the heat capacity in Eq. (30) is positive, which means that the rotating acoustic black hole is thermodynamically stable. Also, it can be easily shown that the curvature scalar of the background geometry, $R = 2[r_H^2 + (J/2)^2]/r^4$, is positive everywhere.

ACKNOWLEDGMENTS

We would like to thank G. Kang for useful discussions. W. Kim and E. J. Son were supported by the Science Research Center Program of the Korea Science and Engineering Foundation through the Center for Quantum Spacetime **(CQUeST)** of Sogang University with grant number R11 - 2005 - 021. Y.-J. Park was supported by the Korea Research Foundation Grant funded by the Korean Government (KRF-2005-015-C00105). M. S. Yoon was supported by the Korea Research Foundation Grant funded by the Korean Government (MOEHRD) (KRF-2005-037-C00017).

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