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Generating Spatial Curvature in an Inhomogeneous Universe: A Bottom-up Approach to Cosmology

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We take the first steps towards a new *bottom-up* approach to cosmology. The dynamics is described in terms of the world lines of the cosmic grains (galaxies or clusters of galaxies). The description is microscopic in the sense that there is no fluid assumption. The motion of each grain is geodesic ensuring the presence of gravitational interactions only. The scheme is fully general in that there is no restriction to homogeneous or isotropic models. Our approach is mathematically similar to Buchert's averaging method, but there are important differences. In particular, we use statistical averages, when needed, not volume averages. For example, a crucial ingredient in any cosmological framework is the spatial curvature. Here we give an estimate of the scalar curvature based on statistical averages of the actual mass distribution in the universe.

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I. INTRODUCTION

Although the Big Bang cosmology is firmly rooted in modern cosmology, the very nature of the matter that the universe is made of has become one of the biggest mysteries in the history of physics and astronomy (but see Ref. 1 for an alternative view). It is fair to say, in our opinion, that we are at an impasse. It is impossible to judge at this time what the best way forward should be when it comes to solving the pressing problems of cosmology. Maybe there is something in the very foundations of cosmology that needs to be reconsidered.

In this note we make a first attempt to reappraise the dust approximation, which has been one of the pillars of cosmology ever since the inception of general relativity almost a hundred years ago. In recent years, the dust paradigm has been challenged by including a general perfect fluid as a source in the field equations (see, *e.g.*, Ref. 2). We will instead challenge the fluid description itself. Another pillar has been the Friedmann (or FLRW) models of a completely homogeneous and isotropic universe. This has also been under much discussion recently as the problem of averaging or back-reaction of local inhomogeneities (see, *e.g.*, Ref. 3 and references therein). This is an important issue and is part of our approach, although we are going one step further by doing away with the fluid description altogether.

The new route to cosmology that we propose is entirely within the framework of general relativity. Our basic philosophy is to use only observed properties of the universe coupled with a minimum of additional assumptions. In particular, we consider only standard general relativity and ordinary matter, possibly augmented by a cosmological constant.

II. GENERAL COSMOLOGICAL CONSIDERATIONS

In this section, we will sketch a cosmological framework within general relativity, which, in particular, will illustrate the role of the spatial curvature. The framework will not be complete in the sense that it will not lead

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directly to a metric.[1] Instead, we will formulate an equation for the scale factor, which may be viewed as a generalization of the Friedmann equation and which is valid in full generality within some weak assumptions. Our approach is related to, but is distinct from, Buchert's averaging scheme [4].

The usual procedure in cosmology is to start with the Einstein equations

$$G_{\alpha\beta} + \Lambda g_{\alpha\beta} = \kappa T_{\alpha\beta} \tag{1}$$

where $G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta}$ and $\kappa = 8\pi G$ with G being Newton's gravitational constant. Assuming isotropy and homogeneity leads directly to the Robertson-Walker metric

$$g_{\rm RW} = -\mathrm{d}t^2 + a(t)^2 \left(\frac{\mathrm{d}r^2}{1-kr^2} + r^2 \mathrm{d}\Omega^2 \right) . \tag{2}$$

The next step in the traditional procedure is to assume that the matter can be described as a zero-pressure fluid (dust); then the Friedmann equation follows. By adopting the assumption of a fluid stress-energy tensor $T_{\mu\nu} = \rho u_\mu u_\nu$, one is effectively smearing the matter in a continuous and homogeneous fashion throughout space. Therefore, the traditional approach to cosmology contains an element of averaging at its very foundation. In fact, as was noted a long time ago by Shirokov and Fisher [5], the cosmological field equations have a hybrid character. They noted that while the left-hand side is microscopic[2] in nature, the right-hand side is, instead, formulated in macroscopic terms. Moreover, the use of a nonzero $T_{\mu\nu}$ is in itself a resort to a non-gravitational ingredient in the theory. It is a basic part of general relativity that purely gravitational interactions should be described by the left-hand side of the Einstein equations, *i.e.*, by the Einstein tensor. It seems that already at the foundation of the usual way of doing cosmology, there are two discrepancies in the relation to the real universe. The first is that the matter is considered as continuous, rather than discrete which would be more realistic. The second is that the matter in the universe is treated as having non-gravitational aspects rather than being regarded as a purely gravitational sector of the theory. Clearly, these discrepancies have been tolerated as necessary compromises in order to find an easy solution to the cosmological problem. However, the price for these compromises has gone up quite sharply in recent years as cosmological observations have become better and better. To explain the apparent acceleration, for example, many authors invoke nonstandard fields that have not been observed in the laboratory or elsewhere (cf. also Ref. 1).

Let us see what can be done without sacrificing a reasonably close connection to the real universe. First of all, we must recognize that an overwhelming fraction of the universe is vacuum. In terms of volume, the interstellar space is about 10^{30} times larger than the interiors of stars and planets.[3] We may take this fact as evidence that the microscopic cosmological Einstein equations should actually be the vacuum equations

$$R_{\mu\nu} = 0 \tag{3}$$

possibly with a cosmological constant added. It might seem that we have then thrown out all matter and that the universe is empty. However, this is not so. A vacuum spacetime need not be empty. The simplest example is the Schwarzschild solution. It is a vacuum solution, but it has matter as a source. Naturally, a cosmological vacuum solution (with matter) differs from the Schwarzschild spacetime in several respects. It should contain a large number of masses, not just one. It must necessarily be nonstationary. Also, the spacetime of the universe is not expected to be asymptotically flat. While the universe is typically assumed to be homogeneous in the large, the Schwarzschild solution certainly is not. These differences can be described collectively in the following way: The cosmological and Schwarzschild spacetimes are both solutions of Eq. (3), but they should satisfy very different boundary conditions. We are using the term boundary conditions here in a broad sense including symmetry requirements.

It is appropriate at this point to discuss the relation to the traditional approach. The fact that the interstellar space is vacuum means that its gravitational field is completely described by the Weyl tensor, $R_{\mu\nu\lambda\sigma} = C_{\mu\nu\lambda\sigma}$. By contrast, a Robertson-Walker metric is conformally flat and hence the Weyl tensor is exactly zero, $C_{\mu\nu\lambda\sigma} = 0$. In this sense, the Robertson-Walker metric is "orthogonal" to any realistic microscopic cosmological metric, which should instead have a vanishing Ricci tensor.

Now suppose we impose exact isotropy and homogeneity as boundary conditions for the universe. Then, as is well-known, the only solution of Eq. (3) is the Minkowski spacetime. This seemingly trivial result has two important consequences. First, it shows that there is no purely gravitational[4] solution in which the matter is continuously distributed in an isotropic and continuous manner. There must be inhomogeneity and/or anisotropy in the matter distribution. This feature is consistent with the Jeans gravitational instability, which became important

[1] However, one should be aware that the metric itself is actually not used for comparison with observations. What is needed is rather the luminosity-redshift relation, which can be derived without using a metric under certain conditions (work in preparation).

[2] We are using the term microscopic here in the sense of referring to the cosmic grains (typically galaxies or galaxy clusters) as microscopic objects.

[3] The vacuum part will be reduced if interstellar matter is taken into account, but will still be large. In any case, there seems to be a consensus that the bulk matter in the universe is moving on geodesics, barring exotic dark energy hypotheses.

[4] Meaning $T_{\mu\nu} = 0$, ignoring here for simplicity a cosmological constant.

after decoupling. If the universe were to be kept in an isotropic and homogeneous state by non-gravitational interactions before decoupling, it couldn't stay that way if the non-gravitational forces were to be switched off. Secondly, it follows that since the universe seems to be approximately isotropic and homogeneous, the cosmological spacetime should in some (although not very well defined) sense be close to Minkowski on the average in a fairly large local region.

Finding realistic exact cosmological solutions to Eq. (3) is very difficult. However, even if it is not be possible to derive a cosmological metric, we can still make progress, as will be described in the next section.

III. BOTTOM-UP COSMOLOGY

Traditionally, an isotropic and homogeneous spacetime is taken as the starting point for cosmology. Sometimes, to broaden the scope, specific inhomogeneous solutions of the Einstein equations are used (*e.g.*, LTB models). Also, a particular stress-energy tensor is postulated, most commonly dust-like matter together with a cosmological constant. To make the models more realistic, some perturbation scheme can then be imposed. Since small perturbations are inadequate to describe the matter distribution in the universe, it has become popular in recent years to use some kind of averaging to allow for large perturbations, which are then said to cause back-reaction effects [4,6]. In short, the starting point in the traditional way of doing cosmology is to impose some large-scale description of the universe and then to compute its evolution and small scale structure. One may refer to this philosophy as a top-down approach.

Instead, our approach is to start from the small scale structure as it is actually observed and then to derive the evolution and large scale observable properties. We refer to this philosophy as *bottom-up cosmology.* In this contribution, we restrict attention to the late-time universe, assuming that gravity alone is responsible for the large scale evolution of the universe.

As a first step in this approach, consider a congruence u^α of observers comoving with the bulk matter. Comparing with Refs. 4, 7, and 8, the difference is that we do not aim at a fluid description. Therefore, our observers (or cosmic grains) should correspond to actual (clusters of) galaxies, not average ones. We may think of these observers as sitting on galaxies that are all moving ballistically. That is, they are only influenced by gravitational forces. This implies that their world lines are geodesics. The volume expansion is defined by $\theta = u^\alpha{}_{;\alpha}$ and is related to the linear Hubble expansion by $H = \theta/3$.

We shall assume that the universe is non-rotating. Given the near isotropy of the CMB radiation, this seems to be a reasonable approximation (cf. Ref. 9). It is then natural to take the congruence u^α to be irrotational. Mathematically, this means that the vorticity is zero; $\omega_{\alpha\beta} = u_{[\alpha;\beta]} = 0$. This is the case precisely if the congruence u^α is hypersurface orthogonal. Then, there exists a foliation $\mathcal{S}_t$ corresponding to surfaces of constant proper time, t, with respect to the observer family u^α. The cosmological scale factor is defined via the relation $\dot{a}/a = H$, where the dot means differentiation with respect to the proper time t.

1. The Generalized Friedmann Acceleration Equation

Given a non-rotating congruence, the evolution of the expansion (or the scale factor) is governed by the Raychaudhuri equation (see, *e.g.*, Ref. 8)

$$\dot{\theta} + \tfrac{1}{3}\theta^2 = 3\frac{\ddot{a}}{a} = -2\sigma^2 - R_{\alpha\beta}u^\alpha u^\beta \qquad (4)$$

where $\sigma^2 = \frac{1}{2}\sigma_{\alpha\beta}\sigma^{\alpha\beta} \geq 0$ is the shear scalar. The shear tensor itself is defined by

$$\sigma_{\alpha\beta} = u_{\mu;\nu}h^\mu{}_\alpha h^\nu{}_\beta - \tfrac{1}{3}\theta h_{\alpha\beta} \qquad (5)$$

where $h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$. The spatial curvature scalar $\mathcal{R}$ of the hypersurfaces $\mathcal{S}_t$ can be expressed in the form of the Gauss-Codazzi equation as [10]

$$\mathcal{R} = -\tfrac{2}{3}\theta^2 + 2\sigma^2 + 2G_{\alpha\beta}u^\alpha u^\beta \ . \qquad (6)$$

Eliminating σ from Eqs. (4) and (6) gives

$$\dot{\theta} + \theta^2 + \mathcal{R} = \frac{3\ddot{a}}{a} + \frac{2\dot{a}^2}{3a^2} + \mathcal{R} = 2G_{\alpha\beta}u^\alpha u^\beta - R_{\alpha\beta}u^\alpha u^\beta \ . \quad (7)$$

This equation can be regarded as a generalized Friedmann acceleration equation. It is valid for any cosmological model under the assumptions stated above. For pressureless matter with a cosmological constant, we can write Eq. (7) as

$$\frac{3\ddot{a}}{a} + \frac{6\dot{a}^2}{a^2} + \mathcal{R} = 12\pi G\rho + 3\Lambda \ . \qquad (8)$$

The relations Eqs. (4) and (8), together with the conservation equation $\dot{\rho} = \theta\rho$ for the energy density, imply an integrability condition for Eq. (8) given by (cf. Ref. 4)

$$\dot{\mathcal{R}} + 2H\mathcal{R} = \dot{\Sigma} + 6H\Sigma \qquad (9)$$

where we have defined $\Sigma := \frac{1}{2}\sigma^2$. It can be written in an equivalent way as

$$a^4\partial_t\big(a^2\mathcal{R}\big) = \partial_t\big(a^6\Sigma\big) \ . \qquad (10)$$

By setting $\sigma = 0$ we can recover the Friedmann equation. In that case, the acceleration equation is not unique because any multiple of Eq. (6) can be added to Eq. (4) without introducing a new unknown. Equations (8) and

(10) have the same form as Buchert's averaged equations. In particular, the system is under-determined, so additional information is needed to obtain solutions. However, our variables are not averages and, therefore, depend on the spatial position. Put another way, each galaxy/observer has its own relations, Eqs. (9) and (10), differing only from others in the specification of initial conditions. The standard way to achieve a closed system is to assume homogeneity and isotropy. Actually, a sufficient condition is to assume that the shear can be neglected. In that case, it follows from Eq. (10) that the spatial curvature will be proportional to a^{-2} as in the FLRW models. In fact, it follows from Eq. (10) that $\mathcal{R} \propto a^{-2}$ if and only if $\sigma \propto a^{-3}$.

Now, imposing vacuum gives the acceleration equation

$$\frac{3\ddot{a}}{a} + \frac{6\dot{a}^2}{a^2} = -\mathcal{R} + 3\Lambda \; . \tag{11}$$

This equation should be integrated together with the integrability condition, Eq. (9). In addition, the evolution of the energy density will be determined by the (macroscopic) conservation equation $\dot{\rho} = \theta\rho$, provided the motion of the matter is non-relativistic.

2. A Toy Model

Let us assume vacuum in accordance with our stated philosophy, $R_{\mu\nu} = 0$, and assume also $\Lambda = 0$. Suppose further that $\mathcal{R} = 6k/a^2$, as in the Friedmann models. Then from (11) we have

$$a\ddot{a} = -2\dot{a}^2 - 2k \; . \tag{12}$$

This equation can be integrated to

$$\frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = \frac{C}{a^6} \tag{13}$$

where C is an integration constant. This equation has the same form as a Friedmann equation corresponding to $p = \rho$, *i.e.*, with stiff matter. However, the interpretation of the right-hand side is different. In the Friedmann case, the right hand-side is proportional to the energy density. To see how C should be interpreted, we refer back to Eq. (6), which shows that

$$\sigma^2 = \frac{3C}{a^6} \; . \tag{14}$$

Therefore, the right-hand side of Eq. (13) should be interpreted as a shear term. The dependence of the shear on the scale factor in Eq. (14) is familiar from anisotropic fluid models (see, *e.g.*, Ref. 8). The relation of C to the deceleration is $\ddot{a} = -2Ca^{-5}$. This can be compared with the Einstein-de Sitter model which has $\ddot{a} \propto a^{-2}$. Furthermore, because our aim is to model a universe that has inhomogeneities, at least locally, the integration constant in Eq. (13) will depend on the particular galaxy world line. Consequently, the inhomogeneities will enter in the initial data in the form of a function $C(x, y, z)$.

Since Eq. (13) does not have the same interpretation as in a FLRW model, we cannot at this stage assess the observational status of the model. Actually, as will be discussed below, the assumption $\mathcal{R} \propto a^{-2}$ does not appear to be realistic.

IV. ESTIMATING THE SPATIAL CURVATURE

The generalized Friedmann equation, Eq. (7), displays clearly the importance of the spatial curvature scalar for the evolution of the cosmological scale factor. However, as noted above, the equations Eqs. (7) and (8) do not form a closed system. On the other hand, the spatial curvature is closely related to the matter content of the universe. In a separate note [11], we discussed the relation between matter and spatial curvature. One result was that the spatial curvature at a given point depends on the velocities of the surrounding matter in addition to mass and position. Matter at rest gives no contribution to the spatial curvature. This latter point is most simply exemplified by the Schwarzschild solution for which the surfaces of constant Killing time (corresponding to the proper time of static observers) have exactly zero spatial curvature [12].

The spatial curvature generated by a single non-relativistic object moving with velocity v in the z-direction at radial distance $r \gg M$ has the form [11]

$$\mathcal{R} = \frac{v^2 M^2}{r^4}(9 + 7\cos 2\theta) \; . \tag{15}$$

By summing the (small) curvatures of individual galaxies, taking into account their peculiar velocities, one obtains

$$\mathcal{R}_{\rm tot} \sim 300\,\bar{v}_p^2 \bar{M}^{2/3} \rho^{4/3} \tag{16}$$

where $\bar{M}$ is the average mass of a typical cosmic grain, ρ is the mass density and $\bar{v}_p$ is the rms average of the peculiar velocities of the cosmic grains. Taking $\bar{M} = 10^{12} M_\odot$ and $\bar{v}_p = 600$ km/s and using $\rho = 0.26\rho_c$ and $H_0 = 71$ km·s^{-1}·Mpc^{-1} yields a curvature parameter $\Omega_k \sim 10^{-12}$ at the present time. It is notable that the sign of the spatial curvature is necessarily positive by Eq. (15). This remains true when rotating objects are also included.

V. DISCUSSION AND OUTLOOK

The positive sign of the spatial curvature seemingly contradicts statements in recent works where a negative spatial curvature is often assumed in voids, for example [13,14]. The reason for this discrepancy is probably the use of the top-down approach. In particular, authors are typically using the Friedmann equation to deduce the spatial curvature. However, the Friedmann equation corresponds to a geometry with vanishing Weyl tensor and non-zero Ricci tensor, in stark contradiction to the actual physical situation, in voids in particular, namely, the interstellar vacuum where the gravitational field should be described by a non-zero Weyl tensor.

A more complete program for a bottom-up cosmology must also include observational relations. Here, we only mention that it is quite possible to reach this goal. What is required is to relate the initial data, as for example the function $C(x, y, z)$ in Eq. (13), to the affine parameter along the light beam between ourselves and the observed object. This can be done under certain simplifying assumptions (work in preparation).

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