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Viscosity Effects in Hydrodynamics via AdS/CFT

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We consider holographic descriptions of Bjorken expansion of strongly coupled large-N_c $\mathcal{N}=4$ SYM fluid in the presence of shear viscosity. We find that the requirement of the regularity of the dual geometry makes a constraint on the time dependence of the shear viscosity. The constraint is consistent with the time evolution of the shear viscosity obtained by its temperature dependence.

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I. INTRODUCTION

Finite temperature systems of quarks and gluons are one of the hot subjects from both the theoretical and the experimental point of view. It has been found that the quark-gluon systems created by the RHIC experiment are in the strong coupling region [1]; hence, we need to develop theoretical methods for strongly coupled YM theories. One of the promising analytical frameworks for strongly coupled YM theories is AdS/CFT [2]. Although the YM theory described by the standard AdS/CFT is large-N_c $\mathcal{N}=4$ SYM theory, there are many attempts to construct models closer to QCD [3]. Another important improvement we need to achieve in AdS/CFT is to include time dependence, since the quark-gluon systems in the RHIC experiment are time-dependent expanding systems. Recently, the authors of [4,5] discussed the time dependent YM systems in the context of non-viscous hydrodynamics by using AdS/CFT [1].

The aim of this report is to obtain an insight into gravity dual descriptions of time-dependent YM systems in the presence of shear viscosity. We shall consider the fluid of strongly coupled large-N_c $\mathcal{N}=4$ SYM theory in AdS/CFT, including the viscous effects. It is known that the $\mathcal{N}=4$ SYM systems at finite temperature have non-zero shear viscosity [7], even at the strong coupling limit. It is also pointed out that inclusion of viscous effects in the analyses of real RHIC physics is important, although the observed shear viscosity is small. (See for example, [8,9].) Therefore, inclusion of the viscous effects is natural and important.

The systems we will consider are time-dependent expanding systems based on Bjorken's picture [10]. In Bjorken's model, the system evolves one-dimensional expansion (Bjorken expansion) along the collision axis of the heavy ions, and the fluid of the quarks and gluons has boost symmetry in the so-called central rapidity region [10]. We shall consider only the late-time regime of the Bjorken expansion where the time evolution is slow enough to employ approximations.

The organization of the paper is the following. In Section II, we review the relativistic hydrodynamics of fluids expanding in Bjorken's picture. In Section III, we review the basic framework of the gravity dual and present some results for non-viscous cases obtained in [4]. The main results of the present work will be given in Section IV. We argue that the regularity of the late-time dual geometry gives a constraint on the time dependence of the shear viscosity. We conclude in the final section.

II. RELATIVISTIC HYDRODYNAMICS WITH SHEAR VISCOSITY

In this section, we review the relativistic hydrodynamics in Bjorken's picture [10]. The energy-momentum tensor in the framework of relativistic hydrodynamics is known to be [2]

$$T^{\mu\nu}=(\rho+P)u^{\mu}u^{\nu}+Pg^{\mu\nu}+\tau^{\mu\nu}, \tag{1}$$

where ρ, P are the energy density and the pressure of the fluid, and $u^{\mu}=(\gamma,\gamma\vec{v})$ is a four-velocity field in terms of the local fluid velocity $\vec{v}$. $\tau^{\mu\nu}$ is the dissipative term. In a frame where the energy three-flux vanishes, $\tau^{\mu\nu}$ is given

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[1] See also [6] for other attempts to describe non-static processes in RHIC physics.

[2] The convention of the signature of the metric is $(-,+,+,+)$.

in terms of the bulk viscosity ξ and the shear viscosity η by

$$\begin{aligned}\tau^{\mu\nu} &= -\eta(\triangle^{\mu\lambda}\nabla_\lambda u^\nu + \triangle^{\nu\lambda}\nabla_\lambda u^\mu - \frac{2}{3}\triangle^{\mu\nu}\nabla_\lambda u^\lambda) \\ &- \xi\,\triangle^{\mu\nu}\nabla_\lambda u^\lambda, \end{aligned} \tag{2}$$

under the assumption that $\tau^{\mu\nu}$ is of first order in gradients. We have defined the three-frame projector as $\triangle^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu$.

We consider pure $\mathcal{N} = 4$ SYM theory whose energy-momentum tensor is traceless. Now, the trace of the energy-momentum tensor is given by

$$T^\mu_\mu = -\rho + 3P - 3\xi\nabla_\lambda u^\lambda. \tag{3}$$

Demanding $T^\mu_\mu = 0$ for all the possible frames where (2) is valid, we obtain

$$\xi = 0, \quad \text{and} \quad \rho = 3P. \tag{4}$$

Notice that the bulk viscosity in the realistic RHIC setup is also negligible. (See, for example, [9].)

We assume that our fluid system is boost-invariant following Bjorken [10], since it is actually supported by experiments. We want to take a "co-moving frame" where each point of the fluid labels the coordinate. In this frame, all the fluid points are at rest by definition; hence, all the fluid points share the same propertime. We can use the rapidity of each fluid point as a spatial coordinate and the common propertime of each fluid point as a time coordinate. Therefore, a local rest frame (LRF) of the fluid can be given by propertime (τ)-rapidity (y), whose relationship with the cartesian coordinate is $(x^0, x^1, x^2, x^3) = (\tau\cosh y, \tau\sinh y, x^2, x^3)$. We have chosen the collision axis to be in the x^1 direction.

The Minkowski metric in this coordinate has the form

$$ds^2 = -d\tau^2 + \tau^2 dy^2 + dx_\perp^2, \tag{5}$$

where $dx_\perp^2 = (dx^2)^2 + (dx^3)^2$. We assume that the collision happened at $\tau = 0$ and we consider only the $\tau \geq 0$ region. Since we are interested in the central rapidity region, we idealize our system by assuming that the fluid is homogeneous in the x^2, x^3 directions and we have boost invariance (in the y direction).

The four-velocity of the fluid at any point in the LRF is $u^\mu = (1, 0, 0, 0)$, and this causes the energy-momentum tensor to be diagonal:

$$T^{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & \frac{1}{\tau^2}\left(P - \frac{4}{3}\frac{\eta}{\tau}\right) & 0 & 0 \\ 0 & 0 & P + \frac{2}{3}\frac{\eta}{\tau} & 0 \\ 0 & 0 & 0 & P + \frac{2}{3}\frac{\eta}{\tau} \end{pmatrix}. \tag{6}$$

We have three independent quantities: ρ, P, and η in (6). However, energy-momentum conservation, $\nabla_\mu T^{\mu\nu} = 0$, together with the equation of state, $\rho = 3P$, reduces the number of the independent quantities to one. One finds that the energy-momentum tensor is written by using only ρ in the following way:

$$T^{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & \frac{1}{\tau^2}\left(-\rho - \tau\dot\rho\right) & 0 & 0 \\ 0 & 0 & \rho + \frac{1}{2}\tau\dot\rho & 0 \\ 0 & 0 & 0 & \rho + \frac{1}{2}\tau\dot\rho \end{pmatrix}, \tag{7}$$

where $\dot\rho \equiv \frac{d\rho}{d\tau}$. By identifying (6) with (7), we obtain the following differential equation that connects η and ρ [3]:

$$\frac{d\rho}{d\tau} = -\frac{4}{3}\frac{\rho}{\tau} + \frac{4}{3}\frac{\eta}{\tau^2}. \tag{8}$$

Note that both ρ and η depend in general on the propertime τ.

Let us consider non-viscous cases ($\eta = 0$). The solution of (8) is then given by

$$\rho(\tau) = \frac{\rho_0}{\tau^{4/3}}, \tag{9}$$

where ρ_0 is a positive constant. The propertime dependence of the temperature T can be read off from Stefan-Boltzmann's law, $\rho \propto T^4$:

$$T = \frac{T_0}{\tau^{1/3}}. \tag{10}$$

This is so-called Bjorken's scaling for the perfect-fluid case [10].

We can evaluate the entropy change. The conservation of energy-momentum tensor and the nature of the one-dimensional expansion give

$$T\frac{d(\tau s)}{d\tau} = 0, \tag{11}$$

where s denotes the entropy density and τs is the entropy per unit rapidity[4]. The entropy per unit rapidity is constant in the absence of viscosity.

III. HOLOGRAPHIC DUAL OF HYDRODYNAMICS

In this section, we review gravity dual analyses of the systems described in the previous section. We consider general asymptotically AdS metrics in the Fefferman-Graham coordinate:

$$ds^2 = r_0^2\frac{g_{\mu\nu}dx^\mu dx^\nu + dz^2}{z^2}, \tag{12}$$

[3] Eq. (8) turns out to be the same as the one appearing in so-called first order (or standard) dissipative relativistic hydrodynamics. (See, for example, Ref. [11] and the references cited therein.) It is known that the first order formalism has a problem of acausal signal propagation. However, we can show that it gives results good enough for our purposes, namely the analyses at the late time.

[4] A precise definition of S is the entropy within a unit 3d region on the (y, x^2, x^3) coordinate. The volume of this region is τ, and it is expanding with time in the x^1 direction.

where $x^\mu = (\tau, y, x^2, x^3)$ in our case. $r_0 \equiv (4\pi g_s N_c \alpha'^2)^{1/4}$ is the length scale given by the string coupling g_s and the number of colors N_c. The four-dimensional metric $g_{\mu\nu}$ is expanded with respect to z in the following form [12,13]:

$$g_{\mu\nu}(\tau, z) = g^{(0)}_{\mu\nu}(\tau) + z^2 g^{(2)}_{\mu\nu}(\tau) + z^4 g^{(4)}_{\mu\nu}(\tau) + \cdots. \quad (13)$$

$g^{(0)}_{\mu\nu}$ is the physical four-dimensional metric for the gauge theory on the boundary, that is given by (5) in the present case. The $g^{(n)}_{\mu\nu}$'s depend only on τ because of the translational symmetry in the x^2, x^3 directions and the boost symmetry in the y direction in our setup. $g^{(2)}_{\mu\nu}$ is found to be zero. We can identify the first non-trivial data in (13), $g^{(4)}_{\mu\nu}$, with the energy-momentum tensor at the boundary [12]:

$$g^{(4)}_{\mu\nu} = \frac{4\pi G_5}{r_0^3} \langle T_{\mu\nu} \rangle, \quad (14)$$

where G_5 is the 5d Newton's constant given by $G_5 = 8\pi^3 \alpha'^4 g_s^2 / r_0^5$ in our notation. We set $4\pi G_5 = 1$ and $r_0 = 1$ in this paper. The higher order terms in (13) are determined by solving Einstein's equation with negative cosmological constant $\Lambda = -6$ [4,12]:

$$R_{MN} - \frac{1}{2} G_{MN} R - 6 G_{MN} = 0, \quad (15)$$

where the metric and the curvature tensor are for the five-dimensional ones of (12). $g^{(2n)}_{\mu\nu}$ is described by $g^{(2n-2)}_{\mu\nu}, g^{(2n-4)}_{\mu\nu}, \cdots, g^{(0)}_{\mu\nu}$ through solving Einstein's equation. In other words, we can obtain the higher order terms in (13) recursively by starting with $g^{(0)}_{\mu\nu}$ ($\sim$Minkowski) and $g^{(4)}_{\mu\nu}$ ($\sim T_{\mu\nu}$).

1. The late-time geometry

The authors of [4] obtained the late-time bulk metric and analyzed the singularity of the late-time geometry. Here, we briefly review their work. The metric is given by

$$\begin{aligned} g_{\tau\tau} &= -1 + \frac{\rho_0}{\tau^{4/3}} z^4 + O(z^6), \\ \frac{g_{yy}}{\tau^2} &= 1 + \frac{\rho_0}{3\tau^{4/3}} z^4 + O(z^6), \\ g_{xx} &= 1 + \frac{\rho_0}{3\tau^{4/3}} z^4 + O(z^6), \end{aligned} \quad (16)$$

where $g_{xx} = g_{22} = g_{33}$. Notice that our Minkowski metric is given by (5). We focus on the late-time behavior of the metric, since $g^{(4)}_{\mu\nu}$ is given only for the late time. This is because slow time evolution is necessary to justify the hydrodynamic treatment of the fluid. If we take the $\tau \to \infty$ limit naively, what we obtain is just the Minkowski metric (5). Therefore, we have to consider more about how to take the late-time limit. To extract a non-trivial result, one needs to take the late-time limit such that $g^{(4)} z^4$ does not go to zero or infinity as $\tau \to \infty$. The authors of [4] found that $g_{\tau\tau}$, g_{yy}/τ^2, and g_{xx} have the following structure by solving Einstein's equation recursively up to a certain order of z:

$$f^{(1)}(v) + f^{(2)}(v)/\tau^{4/3} + \cdots. \quad (17)$$

Therefore, the limit we should take is [4]:

$$\tau \longrightarrow \infty \quad \text{with } \frac{z}{\tau^{1/3}} \equiv v \text{ fixed.} \quad (18)$$

By neglecting the $O(\tau^{-4/3})$ quantities, they obtained an analytic expression of the late-time metric:

$$\begin{aligned} ds^2 = \frac{1}{z^2} \Bigg\{ &-\frac{(1 - \frac{\rho_0}{3}\frac{z^4}{\tau^{4/3}})^2}{1 + \frac{\rho_0}{3}\frac{z^4}{\tau^{4/3}}} d\tau^2 \\ &+ \left(1 + \frac{\rho_0}{3}\frac{z^4}{\tau^{4/3}}\right)(\tau^2 dy^2 + dx_\perp^2) \Bigg\} + \frac{dz^2}{z^2}. \end{aligned} \quad (19)$$

Interestingly, (19) is an (almost) AdS black hole with time-dependent horizon. The time dependence of the entropy and the Hawking temperature from the metric (19) reproduces Bjorken's results (10) and (11).

In [4], singularity analysis was used to select the physical metric. Namely, starting with energy density in the form of [5]

$$\rho = \frac{\rho_0}{\tau^l}, \quad (20)$$

it is found that the late time bulk geometry [6] is singular, except for a special value of l. More precisely, $(R_{MNKL})^2$ at the order of $(\tau)^0$ has a singularity at the horizon, except for

$$l = 4/3, \quad (21)$$

which is the same value obtained at (9).

IV. VISCOUS CASES

1. Viscous hydrodynamics

We now consider the viscous cases. Let us assume that the shear viscosity evolves by

$$\eta = \frac{\eta_0}{\tau^\beta}, \quad (22)$$

where η_0 is a positive constant. The solution of (8) is then given by

$$\rho(\tau) = \frac{\rho_0}{\tau^{4/3}} + \frac{4\eta_0}{1 - 3\beta}\frac{1}{\tau^{1+\beta}} \quad (\text{for } \beta \neq 1/3), \quad (23)$$

where ρ_0 is a positive constant. If $\beta = \frac{1}{3}$, no solution exists. For the $\beta < \frac{1}{3}$ case, the viscous corrections in the

[5] The value of l is restricted to $0 < l < 4$ by the positive energy condition for (7) [4].

[6] In this case, we should replace the v in (18) with $v = z/\tau^{l/4}$.

hydrodynamic quantities become dominant in the late time, which invalidates the hydrodynamic description. If $\beta > \frac{1}{3}$, the shear viscosity is subleading in the late-time behavior, as we expect. We will consider $\beta > \frac{1}{3}$ here.

The propertime dependence of the temperature T is given by Stefan-Boltzmann's law, $\rho \propto T^4$, as:

$$T \sim T_0 \left(\frac{1}{\tau^{1/3}} + \frac{\eta_0}{\rho_0} \frac{1}{1-3\beta} \frac{1}{\tau^\beta} + \cdots \right). \tag{24}$$

In the *static* finite-temperature system of strongly coupled $\mathcal{N} = 4$ SYM theory, it is known that $\eta \propto T^3$ [7]. Let us assume that the same is true in the slowly varying non-static cases. Then we set $\beta = 1$:

$$\eta = \frac{\eta_0}{\tau}. \tag{25}$$

We know $\rho \sim T^4$ and $\eta \sim T^3$ cannot be consistent without an additional term in (25), but the correction term is negligible in our case.

2. Bulk singularity

We have seen that the regularity of the bulk geometry selects the correct time dependence of the energy density in Section III. It is rather interesting that the bulk metric knows the correct form of the energy density independently. Therefore, it is plausible to require the regularity of the geometry at the horizon as a guiding principle; let us see whether it can control the behavior of the viscosity as well. The energy-momentum tensor with generic value of β is:

$$\begin{aligned} T_{\mu\nu} = \mathrm{diag}\Bigg(& \frac{\rho_0}{\tau^{4/3}} + \frac{4\eta_0}{1-3\beta}\frac{1}{\tau^{1+\beta}}, \\ & \tau^2 \left(\frac{\rho_0}{3\tau^{4/3}} + \frac{4\beta\eta_0}{1-3\beta}\frac{1}{\tau^{1+\beta}} \right), \\ & \frac{\rho_0}{3\tau^{4/3}} + \frac{2(1-\beta)\eta_0}{1-3\beta}\frac{1}{\tau^{1+\beta}}, \\ & \frac{\rho_0}{3\tau^{4/3}} + \frac{2(1-\beta)\eta_0}{1-3\beta}\frac{1}{\tau^{1+\beta}} \Bigg), \end{aligned} \tag{26}$$

where we have chosen the leading order behavior of the energy density to be that of the perfect-fluid case.

If $\beta < \frac{1}{3}$, the viscous corrections in $\rho = T_{00}$ are dominant at the late time, and $\rho \sim 1/\tau^{1+\beta}$ from (23). This leads to the singular geometry, since $1 + \beta \neq 4/3$. (See (20) and (21).) Therefore, the region

$$\beta < 1/3 \tag{27}$$

is *excluded* by the regularity requirement. Indeed, our choice $\beta = 1$ is outside the above.

V. CONCLUSIONS

We considered dual geometry of large-N_c $\mathcal{N} = 4$ SYM fluid in the presence of shear viscosity, based on Bjorken's expansion picture. We found that the regularity of the bulk geometry made a constraint of the time dependence of the shear viscosity. The constraint is consistent with the time evolution of the shear viscosity obtained by using its temperature dependence.

If we want to see whether the regularity of the bulk geometry makes further constraints on the shear viscosity, it is necessary to obtain the late-time geometry in the presence of the shear viscosity [14]. It is also important to compute various physical quantities based on the obtained geometry [14].

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